Asymptotic stability of ground states in some Hamiltonian PDEs with symmetry

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Abstract

We consider a ground state (soliton) of a Hamiltonian PDE. We prove that if the soliton is orbitally stable, then it is also asymptotically stable. The main assumptions are transversal nondegeneracy of the manifold of the ground states, linear dispersion (in the form of Strichartz estimates) and nonlinear Fermi Golden Rule. We allow the linearization of the equation at the soliton to have an arbitrary number of eigenvalues. The theory is tailor made for the application to the translational invariant NLS in space dimension 3. The proof is based on the extension of some tools of the theory of Hamiltonian systems (reduction theory, Darboux theorem, normal form) to the case of systems invariant under a symmetry group with unbounded generators.

1 Introduction

In this paper we study the asymptotic stability of the ground state in some dispersive Hamiltonian PDEs with symmetry. We will prove that, in a quite general situation, an orbitally stable ground state is also asymptotically stable. In order to describe the main result of the paper we concentrate on the specific model given by the translationally invariant subcritical NLS in space dimension 3, namely

$$\psi_t = \mathrm{i}\Delta\psi + \mathrm{i}\beta'(|\psi|^2)\psi \ , \quad |\beta^{(k)}(u)| \leq C_k \langle u\rangle^{1+p-k} \ , \quad \beta'(0) = 0 \ . \tag{1.1}$$

$p < \frac{2}{3}$, $x \in \mathbb{R}^3$. It is well known that, under suitable assumptions on β, such an equation has a family of ground states which can travel at any velocity and which are orbitally stable (see e.g. [FGJS04] for a review). Consider the linearization of the NLS at the soliton, and let L_0 be the linear operator describing such a linearized system. Due to the symmetries of the system, zero is always an eigenvalue of L_0 with algebraic multiplicity at least 8. In the case where this is the exact multiplicity of zero and L_0 has no other eigenvalues, asymptotic stability was proven in [BP92, Cuc01] (see also [Per11]). Here we tackle the case where L_0 has an arbitrary number of eigenvalues, disjoint from the essential spectrum, and prove that, assuming a suitable version of the Fermi Golden Rule

(FGR), the ground state is (orbitally) asymptotically stable. We recall that the importance of the FGR in nonlinear PDEs was understood by Sigal [Sig93] and shown to have a crucial role in the study of asymptotic stability in [SW99]. Similar conditions have been used and generalized by many authors. The FGR that we use here is a generalization of that of [GW08] (see also [BC11, Cuc11a]).

The present paper is a direct development of [BC11] and [Cuc11a], which in turn are strongly related to [GS07, GNT04, CM08, GW08]. We recall that in [BC11] Hamiltonian and dispersive techniques were used to prove that the empty state of the nonlinear Klein Gordon equation is asymptotically stable even in the presence of discrete spectrum of the linearized system. Then [Cuc11a] extended the techniques of [BC11] to the study of the asymptotic stability of the ground state in the NLS with a potential.

The main novelty of the present paper is that we deal here with the translational invariant case. The new difficulty one has to tackle is related to the fact that the group of the translations $\psi(.) \mapsto \psi(. - t\mathbf{e}_i)$ is generated by $-\partial_{x_i}$ which is an unbounded operator: it turns out that this obliges to use non smooth maps in order to do some steps of the proof. To overcome this problem we introduce and study a suitable class of maps, that we call "almost smooth" (see in particular sect.3.2). We use them to develop Hamiltonian reduction theory, Darboux theorem and also canonical perturbation theory.

The fact that the generator of the translations is not smooth causes some difficulties also in the use of Strichartz estimates, but such difficulties were already overcome by Perelman [Per11] (see also [Bec11]), so we simply apply her method to our case.

We now describe the proof. First, we use Marsden Weinstein reduction procedure in order to deal with the symmetries. In order to overcome the problems related to the fact that the generators of the symmetry group are unbounded, we fix a concrete local model for the reduced manifold and work in it. The local model is a submanifold contained in the level surface of the integrals of motion. The restriction of the Hamiltonian and of the symplectic form to such a submanifold give rise to the Hamiltonian system one has to study. The advantage of such an approach is that the ground state appears as a minimum of the Hamiltonian, so one is reduced to study the asymptotic stability of an elliptic equilibrium, a problem close to that stidued in [BC11]. However the application of the methods of [BC11, Cuc11a] to the present case is far from trivial, since the restriction of the symplectic form to the submanifold turns out to be in noncanonical form, and to have non smooth coefficients (some "derivatives" appear). So, we proceed by first proving a suitable version of the Darboux theorem which reduces the symplectic form to the canonical one. This requires the use of non smooth transformations. We point out that a key ingredient of our developments is that the ground state is a Schwartz function, and this allows to proceed by systematically moving derivatives from the unknown function to the ground state.

Then we study the structure of the Hamiltonian in the Darboux coordinates

and prove that it has a precise (and quite simple) form. Subsequently, following [BC11, Cuc11a], we develop a suitable version of normal form theory in order to extract the essential part of the coupling between the discrete modes and the continuous ones. Here we greatly simplify the theory of [BC11, Cuc11a]. In particular we think that we succeeded in developing such a theory under minimal assumptions. We also point out that in the present case the canonical transformations putting the system in normal form are not smooth, but again almost smooth.

Finally, following the scheme of [GNT04, CM08, BC11, Cuc11a], we use Strichartz estimates in order to prove that there is dispersion, and that the energy in the discrete degrees of freedom goes to zero as $t \to \infty$. As we already remarked there are some difficulties in the linear theory, difficulties that we overcome using the methods of [Per11]. In this part, we made an effort to point out the properties that the nonlinearity has to fulfill in order to ensure the result. Thus we hope to have proved a result which can be simply adapted to different models.

We now discuss more in detail the relation with the paper [Cuc11a]. In [Cuc11a] Cuccagna studied the case of NLS with a potential and proved a result similar to the present one. Here we generalize Cuccagna's result in several aspects. The first one is that we allow the system to have symmetry groups with more than one dimension, but the main improvement we get consists of the fact that we allow the symmetries to be generated by unbounded operators (as discussed above). Furthermore we work in an abstract framework.

Finally, we work here on the reduced system (according to Marsden-Weinstein theory), but we think that all the arguments developed in such a context could be reproduced also working in the original phase space. We also expect that the same (maybe more) difficulties will appear also when working in the original phase space.

Three days before the first version of this paper was posted in Arxiv, the paper [Cuc11b] was also posted there. The paper [Cuc11b] deals exactly with the same problem. The result of [Cuc11b] is very close to the present one, but weaker: the result of such a paper is valid only for initial data of Schwartz class, while the control of the difference between the soliton and the solution is obtained in energy norm, and no decay rate is provided. Such a kind of conclusions is usual for initial data in the energy space, while the typical result valid for solutions corresponding to initial data decaying in space also controls the rate of decay of the solution to the ground state. On the contrary, in the present paper we give a result valid for any initial datum of finite energy (and of course we do not deduce a decay rate).

A further difference between the two papers is that, here a large part of the proof is developed in an abstract framework, thus we expect our result to be simply applicable also to different systems. We are not aware of other papers in the domain of asymptotic stability in dispersive Hamiltonian PDEs in which the proof is developed in an abstract framework.

Our proof is also much simpler than that of [Cuc11b], indeed in order to generate the flow of the transformation introducing Darboux coordinates (and the transformations putting the system in normal form) we use a techinique coming from the theory of semilinear PDEs, while [Cuc11b] uses techniques coming from quasilinear PDEs.

A further difference, is that we work using Marsden Weinstein reduction, while [Cuc11b] works in the original phase space.

The paper is organized as follows: in sect. 2 we state our main result for the NLS; in sect. 3 we set up the abstract framework in which we work and state and prove the Darboux theorem mentioned above; in sect. 4 we study the form of the Hamiltonian in the Darboux coordinates. In sect. 5 we use canonical perturbation theory in order to decouple as far as possible the discrete degrees of freedom from the continuous ones; in sect. 6 we prove that the variables corresponding to the continuous spectrum decay dispersively and the variables corresponding to the discrete spectrum decay at zero; here the main abstract theorem 6.1 is stated and proved; in sect. 7 we apply the abstract theory to the NLS. In the first Appendix we prove that the dynamics of the reduced system, while in the second one, we reproduce Perelman's Lemma on the dispersion of the linear system.

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2 Asymptotic stability in NLS

We state here our result on the NLS eq. (1.1). We assume

(H1) There exists an open interval $\mathcal{I} \subset \mathbb{R}$ such that, for $\mathcal{E} \in \mathcal{I}$ the equation

$$-\Delta b_{\mathcal{E}} - \beta'(b_{\mathcal{E}}^2) b_{\mathcal{E}} + \mathcal{E} b_{\mathcal{E}} = 0 \; , \tag{2.1}$$

admits a C^∞ family of positive, radially symmetric functions $b_{\mathcal{E}}$ belonging to the Schwartz space.

(H2) One has $\dfrac{d}{d\mathcal{E}} \left\| b_{\mathcal{E}} \right\|_{L^2}^2 > 0, \; \mathcal{E} \in \mathcal{I}$.

Then one can construct traveling solitons, which are solutions of (1.1) of the form

$$\psi(x,t) = e^{-\mathrm{i}\left(\mathcal{E} - \frac{|v|^2}{4}\right)t} \mathrm{e}^{-\mathrm{i}\frac{v\cdot x}{2}} b_{\mathcal{E}}(x - vt) \; . \tag{2.2}$$

(H3) Consider the operators

$$A_+ := -\Delta + \mathcal{E} - \beta'(b_{\mathcal{E}}^2) \ , \quad A_- := -\Delta + \mathcal{E} - \beta'(b_{\mathcal{E}}^2) - 2\beta''(b_{\mathcal{E}}^2)b_{\mathcal{E}}^2 \ , \tag{2.3}$$

then the Kernel of the operator A_+ is generated by $b_{\mathcal{E}}$ and the Kernel of the operator A_- is generated $\partial_j b_{\mathcal{E}}$, $j = 1, 2, 3$.

Remark 2.1. Under the above assumptions the solutions (2.2) are orbitally stable (see e.g. [FGJS04]).

In order to state the assumptions on the linearization at the soliton insert the following Ansatz in the equations

$$\psi(x,t) = e^{-\mathrm{i}\left(\mathcal{E} - \frac{|v|^2}{4}\right)t} e^{-\mathrm{i}\frac{v\cdot x}{2}} \left(b_{\mathcal{E}}(x - vt) + \chi(x - vt)\right) \ , \tag{2.4}$$

and linearize the so obtained equation in χ. Then one gets an equation of the form $\dot{\chi} = L_0\chi$ with a suitable L_0. It can be easily proved that the essential spectrum of L_0 is $\bigcup_{\pm} \pm\mathrm{i}[\mathcal{E}, +\infty)$ and that 0 is always an eigenvalue. The rest of the spectrum consists of purely imaginary eigenvalues $\pm\mathrm{i}\omega_j$, that we order as follows $0 < \omega_1 \le \omega_2 \le ... \le \omega_K$. We assume that

(H4) $\omega_K < \mathcal{E}$. Furthermore, let r_t be the smallest integer number such that $r_t\omega_1 > \mathcal{E}$, then we assume $\omega \cdot k \neq \mathcal{E}$, $\forall k \in \mathbb{Z}^K \ : \ |k| \le 2r_t$.

(H5) $\pm\mathrm{i}\mathcal{E}$ are not resonances of L_0.

(H6) The Fermi Golden Rule (6.52) holds.

The main theorem we are now going to state refers to initial data ψ_0 which are sufficiently close to a ground state. In its statement we denote by ϵ the quantity below

$$\epsilon := \inf_{q_0^4\in\mathbb{R}, \mathbf{q}_0\in\mathbb{R}^3, v_0\in\mathbb{R}^3, \mathcal{E}_0\in\mathcal{I}} \left\| \psi_0 - \mathrm{e}^{-\mathrm{i}q_0^4} \mathrm{e}^{-i\frac{v_0\cdot x}{2}} b_{\mathcal{E}_0}(x - \mathbf{q}_0) \right\|_{H^1} \tag{2.5}$$

Theorem 2.2. *Assume ϵ is small enough, then there exist C^1 functions*

$$\mathcal{E}(t), v(t), q^4(t), \mathbf{q}(t), y^4(t), \mathbf{y}(t) \ ,$$

and $\psi_+ \in H^1$ such that the solution $\psi(t)$ with initial datum ψ_0 admits the decomposition

$$\psi(x,t) = \mathrm{e}^{-\mathrm{i}q^4(t)} \mathrm{e}^{-\mathrm{i}\frac{v(t)\cdot x}{2}} b_{\mathcal{E}(t)}(x - \mathbf{q}(t)) + \mathrm{e}^{-\mathrm{i}y^4(t)}\chi(x - \mathbf{y}(t), t) \tag{2.6}$$

and

$$\lim_{t\to+\infty} \left\|\chi(t) - \mathrm{e}^{\mathrm{i}t\Delta}\psi_+\right\|_{H^1} = 0 \ . \tag{2.7}$$

Furthermore the functions $\mathcal{E}(t), v(t), \dot{q}^4(t), \dot{y}^4(t), \dot{\mathbf{q}}(t), \dot{\mathbf{y}}(t)$ admit a limit as $t \to +\infty$.

The rest of the paper is devoted to the proof of an abstract version of this theorem.

3 General framework and the Darboux theorem

Consider a scale of Hilbert spaces $\mathcal{H}^k$, $k \in \mathbb{Z}$. The scalar product in $\mathcal{H}^0$ will be denoted by $\langle .;. \rangle$; such a scalar product is also the pairing between $\mathcal{H}^k$ and $\mathcal{H}^{-k}$. We will denote $\mathcal{H}^\infty := \cap_k \mathcal{H}^k$, and $\mathcal{H}^{-\infty} := \cup_k \mathcal{H}^k$. Let $E : \mathcal{H}^k \to \mathcal{H}^k$, $\forall k$ be a linear continuous operator skewsymmetric with respect to $\langle .;. \rangle$. Assume it is continuously invertible. Let $J : \mathcal{H}^k \to \mathcal{H}^k$ be its inverse (Poisson tensor). We endow the scale by the symplectic form $\omega(U_1, U_2) := \langle EU_1; U_2 \rangle$, then the Hamiltonian vector field X_H of a function H is defined by $X_H = J\nabla H$,where ∇H is the gradient with respect to the scalar product of $\mathcal{H}^0$.

Remark 3.1. In the application to dispersive equations one has to deal with weighted Sobolev space H^{k_1,k_2}, which are labeled by a couple of indexes. All what follows holds also in such a situation provided one defines the notation $(k_1, k_2) > (l_1, l_2)$ by $k_1 > l_1$ and $k_2 \geq l_2$.

For $j = 1, ..., n$, let $A_j : \mathcal{H}^k \to \mathcal{H}^{k-d_j}$, $\forall k \in \mathbb{Z}$ and some $d_j \geq 0$, be n bounded selfadjoint (with respect to $\langle .;. \rangle$) linear operators, and consider the Hamiltonian function $\mathcal{P}_j(u) := \langle A_j u; u \rangle /2$. Then $X_{\mathcal{P}_j} = JA_j$ generates a flow in $\mathcal{H}^0$ denoted by e^{tJA_j}.

Remark 3.2. In the case of multiple indexes the index d_j represents the loss of smoothness and always acts only on the first index, namely one has $A_j \mathcal{H}^{k_1,k_2} \subset \mathcal{H}^{k_1-d_j,k_2}$.

Remark 3.3. The operators JA_j will play the role of the generators of the symmetries of the Hamiltonian system we will study. Correspondingly the functions $\mathcal{P}_j$ will be integrals of motion.

We denote $d_A := \max_{j=1,...,n} d_j$. For $i, j = 1, ..., n$ we assume that, on $\mathcal{H}^\infty$ one has

(S1) $[A_j, E] = 0$,

(S2) $A_i JA_j = A_j JA_i$ which implies $\{\mathcal{P}_j, \mathcal{P}_i\} = 0 = \langle A_j u; JA_i u \rangle$.

(S3) For any $t \in \mathbb{R}$ the map e^{tJA_j} leaves invariant $\mathcal{H}^\infty$.

Let A_0 be a linear operator with the same properties of the A_j's. Assume $d_0 \geq d_A$. The Hamiltonian we will study has the form

$$H(u) = \mathcal{P}_0(u) + H_P(u) \ , \quad \mathcal{P}_0(u) = \frac{1}{2}\langle u; A_0 u \rangle \ , \tag{3.1}$$

where H_P is a nonlinear term on which we assume

(P1) There exists k_0 and an open neighborhood of zero $\mathcal{U}^{k_0} \subset \mathcal{H}^{k_0}$ such that $H_P \in C^\infty(\mathcal{U}^{k_0}, \mathbb{R})$.

We also assume that (on $\mathcal{H}^\infty$)

(S4) H_P and $\mathcal{P}_0$ Poisson commutes with each one of the functions $\mathcal{P}_j$:

$$\{\mathcal{P}_0; \mathcal{P}_j\} = \{H_P; \mathcal{P}_j\} = 0 \ , \quad j = 1, ..., n \tag{3.2}$$

We are interested in bound states η, namely in phase points such that $u(t) := e^{t\lambda^j JA_j}\eta$ is a solution of the Hamilton equations of H. *Here and below we use Einstein notation according to which sum over repeated indexes is understood.* The indexes will always run between 1 and n. Then η has to fulfill the equation

$$A_0\eta + \nabla H_P(\eta) - \lambda^j A_j \eta = 0 \ . \tag{3.3}$$

We assume

(B1) There exists an open set $I \subset \mathbb{R}^n$ and a C^∞ map

$$I \ni p \mapsto (\eta_p, \lambda(p)) \in \mathcal{H}^\infty \times \mathbb{R}^n \ ,$$

s.t. $(\eta_p, \lambda(p))$ fulfills equation (3.3). Furthermore the map $p \to \lambda$ is 1 to 1.

(B2) For any fixed $p \in I$, the set $\mathcal{C} := \bigcup_{q\in\mathbb{R}^n} e^{q^j JA_j}\eta_p$ is a smooth n dimensional submanifold of $\mathcal{H}^\infty$.

(B3) The manifold $\bigcup_{p\in I}\eta_p$ is isotropic, namely the symplectic form ω vanishes on its tangent space.

By (B1) it is possible to normalize the values of p_j in such a way that $\mathcal{P}_j(\eta_p) = p_j$, *Form now on we will always assume such a condition to be satisfied.*

Remark 3.4. By the proof of Arnold Liouville's theorem, the manifold $\mathcal{C}$ of hypothesis (B2) is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Consider the symplectic manifold

$$\mathcal{T} := \bigcup_{q\in\mathbb{R}^n, p\in I} e^{q^j JA_j}\eta_p \ ,$$

namely the manifold of bound states; its tangent space is given by

$$T_{\eta_p}\mathcal{T} := \text{span}\left\{ JA_j\eta_p, \frac{\partial \eta_p}{\partial p_j} \right\} \tag{3.4}$$

and its symplectic orthogonal $T^\omega_{\eta_p}\mathcal{T}$ is given by

$$T^\omega_{\eta_p}\mathcal{T}$$
$$= \left\{ U \in \mathcal{H}^{-\infty} \ : \omega\left(JA_j\eta_p; U\right) = \langle A_j\eta_p; U\rangle = \omega\left(\frac{\partial \eta_p}{\partial p_j}; U\right) = \langle E\frac{\partial \eta_p}{\partial p_j}; U\rangle = 0 \right\}$$

Lemma 3.5. *One has $\mathcal{H}^{-\infty} = T^\omega_{\eta_p}\mathcal{T} \oplus T_{\eta_p}\mathcal{T}$. Explicitly the decomposition of a vector $U \in \mathcal{H}^{-\infty}$ is given by*

$$U = P_j \frac{\partial \eta_p}{\partial p_j} + Q^j JA_j\eta_p + \Phi_p \ , \tag{3.5}$$

with

$$Q^j = -\langle E\frac{\partial \eta_p}{\partial p_j}; U\rangle \ , \qquad P_j = \langle A_j \eta_p; U\rangle \tag{3.6}$$

and $\Phi_p \in T^{\omega}_{\eta_p}\mathcal{T}$ *given by*

$$\Phi_p = \Pi_p U := U - \langle A_j\eta_p; U\rangle \frac{\partial \eta_p}{\partial p_j} + \langle E\frac{\partial \eta_p}{\partial p_j}; U\rangle J A_j \eta_p \ . \tag{3.7}$$

Proof. The first of (3.6) is obtained taking the scalar product of (3.5) with $-E\frac{\partial \eta_p}{\partial p_j}$, and exploiting

$$\langle E\frac{\partial \eta_p}{\partial p_j}, \frac{\partial \eta_p}{\partial p_k}\rangle = 0 \tag{3.8}$$

which is equivalent to (B3). Taking the scalar product of (3.5) with $A_j\eta_p$ we get the second of (3.6). Then (3.7) immediately follows. $\square$

Remark 3.6. A key point in all the developments of the paper is that the projector Π_p defined by (3.7) is a smoothing perturbation of the identity, namely $\mathbb{1} - \Pi_p \in C^\infty(I, B(\mathcal{H}^{-k}, \mathcal{H}^l))$, $\forall k, l$, where $B(\mathcal{H}^{-k}, \mathcal{H}^l)$ is the space of bounded operators from $\mathcal{H}^{-k}$ to $\mathcal{H}^l$. In particular one has that $\frac{\partial \Pi_p}{\partial p_i} \in C^\infty(I, B(\mathcal{H}^{-k}, \mathcal{H}^l))$

An explicit computation shows that the adjoint of Π_p is given by

$$\Pi_p^* U := U - \langle \frac{\partial \eta_p}{\partial p_j}; U\rangle A_j\eta_p + \langle J A_j \eta_p, U\rangle E\frac{\partial \eta_p}{\partial p_j} \ . \tag{3.9}$$

Some useful formulae are collected below

$$E\Pi_p = \Pi_p^* E \ , \quad J\Pi_p^* = \Pi_p J \ , \quad \frac{\partial \Pi_p}{\partial p_j} = \frac{\partial \Pi_p^2}{\partial p_j} = \Pi_p \frac{\partial \Pi_p}{\partial p_j} + \frac{\partial \Pi_p}{\partial p_j}\Pi_p \ . \tag{3.10}$$

$$\Pi_p \frac{\partial \Pi_p}{\partial p_j}\Pi_p = 0 \ , \quad \left(\frac{\partial \Pi_p}{\partial p_j}\right)^* = \frac{\partial \Pi_p^*}{\partial p_j} \ , \quad E\frac{\partial \Pi_p}{\partial p_j} = \frac{\partial \Pi_p^*}{\partial p_j} E \ . \tag{3.11}$$

In the following we will work locally close to a particular value $p_0 \in I$**. Thus we fix** it and define

$$\mathcal{V}^k := \Pi_{p_0}\mathcal{H}^k \tag{3.12}$$

which we endow by the topology of $\mathcal{H}^k$. Similarly we define $(\mathcal{V}^k)^* := \Pi_{p_0}^*\mathcal{H}^k$. When we do not put an exponent we mean $\mathcal{V}^0$.

Remark 3.7. For any positive k, l, one has

$$\|(\Pi_p\Pi_{p'} - \Pi_{p'})u\|_{\mathcal{H}^k} \le C_{k,l}\,|p - p'|\,\|u\|_{\mathcal{H}^{-l}} \ , \tag{3.13}$$

and, by the first of (3.11), for $\phi \in \mathcal{V}^\infty$, one has

$$\left\|\Pi_p\frac{\partial \Pi_p}{\partial p_j}\phi\right\|_{\mathcal{H}^k} \le C_{kl}\,|p - p_0|\,\|\phi\|_{\mathcal{H}^{-l}} \ . \tag{3.14}$$

Remark 3.8. Consider the operator $\Pi_p : \mathcal{V}^{-\infty} \to \Pi_p \mathcal{H}^{-\infty}$; it has the structure $1\!\!1 + (\Pi_p - \Pi_{p_0})$, and one has

$$\left\| (\Pi_p - \Pi_{p_0})\phi \right\|_{\mathcal{H}^k} \leq C\, |p - p_0|\, \|\phi\|_{\mathcal{H}^{-l}} \ . \tag{3.15}$$

Thus, by Neumann formula the inverse $\widetilde{\Pi_p}^{-1}$ of Π_p has the form $\widetilde{\Pi_p}^{-1} = 1\!\!1 + S$ with S sulfilling (3.15).

3.1 Reduced manifold

We introduce now the reduced symplectic manifold obtained by exploiting the symmetry. In the standard case where the generators of the symmetry group are smooth (i.e. $d_j = 0$) the construction is standard and goes as follows.

Fix $p_0 \in I$ as above and define a surface $\mathcal{S} = \{u \ : \ \mathcal{P}_j(u) = p_{0j}\}$, then pass to the quotient with respect to the group action of $\mathbb{R}^n$ on $\mathcal{S}$ defined by $(q, u) \mapsto \mathrm{e}^{q^j J A_j} u$, obtaining the reduced phase space $\mathcal{M}$. A local model of $\mathcal{M}$ close to η_{p_0} is obtained by taking a codimension n submanifold of $\mathcal{S}$ transversal to the orbit of the group. Here we proceed the other way round: we choose a submanifold $\mathcal{M} \subset \mathcal{S}$ of codimension n, transversal to the orbit of the group at η_{p_0}, and we study the Hamiltonian system obtained by restricting the Hamiltonian to $\mathcal{M}$.

Consider the map

$$I \times \mathcal{V} \ni (p, \phi) \mapsto i_0(p, \phi) := \eta_p + \Pi_p \phi \ ; \tag{3.16}$$

we will use the implicit function theorem (see lemma 3.11) in order to compute $p_j = p_j(\phi)$ in such a way that the image of the map

$$\mathcal{V} \ni \phi \mapsto i(\phi) := \eta_{p(\phi)} + \Pi_{p(\phi)} \phi \subset \mathcal{S} \ , \tag{3.17}$$

is the wanted local model of $\mathcal{M}$, and i is a local coordinate system in it. In studying this map we will use a class of maps which will play a fundamental role in the whole paper. In the corresponding definitions we will consider maps from $\mathbb{R}^n \times \mathcal{V}^k$ to some space. *By this we* **always** *mean a map defined in an open neighborhood of the origin. Since the width of the neighborhood does not play any role in the future we avoid to specify it.*

Definition 3.9. A map $S : \mathcal{V}^{d_A/2} \to \mathcal{H}^{\infty}$ will be said to be of class $\mathcal{S}^i_j$ if there exists a smooth map $\tilde{S} : \mathbb{R}^n \times \mathcal{H}^{-\infty} \to \mathcal{H}^{\infty}$ such that $S(\phi) = \tilde{S}(\mathcal{P}(\phi), \phi)$, and the map $\tilde{S}$ fulfills

$$\left\| \tilde{S}(N, \phi) \right\|_{\mathcal{H}^m} \leq C_{mk} |N|^i \, \|\phi\|^j_{\mathcal{H}^{-k}} \ , \quad \forall m, k \geq 0 \ . \tag{3.18}$$

In the case of maps taking values in $\mathbb{R}^n$ we give an analogous definition.

Definition 3.10. A map $R : \mathcal{V}^{d_A/2} \to \mathbb{R}^n$ will be said to be of class $\mathcal{R}^i_j$ if there exists a smooth map $\tilde{R} : \mathbb{R}^n \times \mathcal{H}^{-\infty} \to \mathbb{R}^n$ such that $R(\phi) = \tilde{R}(\mathcal{P}(\phi), \phi)$, and the map $\tilde{R}$ fulfills

$$\left\| \tilde{R}(N, \phi) \right\| \leq C_k |N|^i \, \|\phi\|^j_{\mathcal{H}^{-k}} \ , \quad \forall k \geq 0 \ . \tag{3.19}$$

The functions belonging to the above classes will be called smoothing.

In the following we will identify a smoothing function S (or R) with the corresponding function $\tilde{S}$ (or $\tilde{R}$). Most of the times functions of class $\mathcal{S}_l^k$ ($\mathcal{R}_k^l$ resp.) will be denoted by S_l^k (R_l^k resp.). Furthermore, since the only relevant property of such functions are given by the inequalities (3.18) and (3.19) we will use the same notation for different smoothing functions. For example we will meet equalities of the form

$$S_1^1 + S_2^1 = S_1^1 \tag{3.20}$$

where obviously the function S_1^1 at r.h.s. is different from that at l.h.s.

Finally, we always consider functions and vector fields as functions of N, ϕ, with the idea that, at the end of the procedure we will put $N_j = \mathcal{P}_j(\phi)$.

Lemma 3.11. *There exists a smoothing map $p \in \mathcal{R}_0^0$ with the following properties*

(1) For any $j = 1, ..., n$, and for $\phi \in \mathcal{V}^{d_A/2}$, one has

$$\mathcal{P}_j(\eta_{p(\mathcal{P}(\phi),\phi)} + \Pi_{p(\mathcal{P}(\phi),\phi)}\phi) = p_{0j} \ ;$$

(2) there esist $R_2^1 \in \mathcal{R}_2^1$ s.t. $p = p_0 - N + R_2^1(N, \phi)$;

(3) Define the matrix $M = M(N, \phi)$, by $(M^{-1})_{jk} = \delta_{jk} + \langle \Pi_p\phi; A_j \frac{\partial \Pi_p}{\partial p_k}\phi\rangle$ (evaluated at $p = p(N, \phi)$), then the gradient of $p(\mathcal{P}(\phi), \phi)$ is given by

$$\nabla p_j = -\sum_k M_{jk}\Pi_{p_0}^* A_k\phi \ . \tag{3.21}$$

Proof. First remark that one has

$$\mathcal{P}_j(\eta_p + \Pi_p\phi) = p_j + \mathcal{P}_j(\Pi_p\phi) = p_j + \mathcal{P}_j(\phi) + V_j(p, \phi) \ , \tag{3.22}$$

where $V_j(p, \phi) := \mathcal{P}_j(\Pi_p\phi) - \mathcal{P}_j(\phi)$ extends to a smooth map on $\mathcal{V}^{-k}$ $\forall k$ and fulfills

$$|V_j(p, \phi)| \leq C_k\, |p - p_0|\, \|\phi\|^2_{\mathcal{H}^{-k}} \ , \quad \forall k \tag{3.23}$$

We apply the implicit function theorem to the system of equations

$$0 = F_j(p, N, \phi) := p_j + N_j + V_j(p, \phi) - p_{0j} \ . \tag{3.24}$$

Using (3.22) one gets

$$\frac{\partial F_j}{\partial p_k} = \delta_j^k + \langle \Pi_p\phi; A_j \frac{\partial \Pi_p}{\partial p_k}\phi\rangle = (M^{-1})_{jk} \ .$$

Since $\frac{\partial F_j}{\partial p_k} \equiv (M^{-1})_{jk}$ is invertible, the implicit function theorem ensures the existence of a smooth function $p = p(N, \phi)$ from $\mathbb{R}^n \times \mathcal{H}^{-k}$ to $\mathbb{R}^n$ solving (3.24). Then the estimate ensuring $p - (p_0 - N) \in \mathcal{R}_2^1$ follows from the fact that $p(p_0, \phi) -$

$(p_0 - N) = 0$ and from the computation of the differential of p with respect to ϕ, which gives $\mathrm{d}_\phi p(N, \phi) = -M \mathrm{d}_\phi V$, which in turn shows that $\mathrm{d}_\phi p(N, 0) = 0$ since V is quadratic in ϕ. Equation (3.21) is an immediate consequence of the formula for the derivative of the implicit function applied directly to (3.22) $= p_0$. $\square$

We are now going to study the correspondence between the dynamics in $\mathcal{V}$ and the dynamics of the complete system. We endow $\mathcal{V}$ by the symplectic form $\Omega := i^*\omega$ (pull back) and consider an invariant Hamiltonian function $H : \mathcal{H}^k \to \mathbb{R}$, namely a function with the property that $H(e^{q^j J A_j} u) = H(u)$. We will denote by $H_r := i^* H$ the corresponding reduced Hamiltonian.

Remark 3.12. By the smoothness of the map $p \mapsto \eta_p$, there exists a map $S_1^1 \in \mathcal{S}_1^1$ such that $H_r(\phi) = H(\eta_{p(N,\phi)} + \Pi_{p(N,\phi)}\phi) = H(\eta_{p_0} + \phi + S_1^1)$, a formula which will be fundamental in the following.

The function H_r defines a Hamiltonian system on $\mathcal{V}$. We will denote by X_{H_r} the corresponding Hamiltonian vector field. Denote also by X_H the Hamiltonian vector field of H in the original phase space. Before stating the theorem on the correspondence of the solutions we specify what we mean by solution.

Definition 3.13. Let k be fixed. A function $u \in C^0([0,T]; \mathcal{H}^k)$ will be said to be a solution of $\dot{u} = X(u)$, if there exists a sequence of functions $u_l \in C^1([0,T]; \mathcal{H}^k)$ which fulfill the equation and converge to u in $C^0([0,T]; \mathcal{H}^k)$.

Theorem 3.14. *Assume that X_{H_r} defines a local flow in $\mathcal{V}^l$ for some $l \geq 0$. Assume that such a flow leaves invariant $\mathcal{V}^k$ for some k large enough. Let $u_0 := e^{q_0^j J A_j} i(\phi_0)$, be an initial datum with $\phi_0 \in \mathcal{V}^l$. Consider the solution $\phi(t) \in \mathcal{V}^l$ of the Cauchy problem $\dot{\phi} = X_{H_r}(\phi)$, $\phi(0) = \phi_0$. Then there exist C^1 functions $q(t)$ such that*

$$u(t) := e^{q^j(t) J A_j} i(\phi(t)) \tag{3.25}$$

is a solution of $\dot{u} = X_H(u)$ with initial datum u_0. Viceversa, if X_H generates a local flow, for initial data close to $\mathcal{C}$ and such a flow leaves invariant $\mathcal{H}^k$ for some k large enough, then any solution of the original system admits the representation (3.25)*, with $\phi(t)$ a solution of the reduced system.*

The proof is obtained more or less as in the standard way (see e.g. [Sch87]), however one has to verify that all of the formulae that are used keep a meaning also in the present non smooth case, and this is quite delicate. For this reason the proof is deferred to the appendix A.

3.2 Almost smooth maps and the Darboux theorem

Denote $\Omega := i^*\omega$. By construction it is clear that

$$\Omega\big|_{\phi=0} (\Phi_1; \Phi_2) = \langle E\Phi_1; \Phi_2 \rangle \ . \tag{3.26}$$

We will transform the coordinates in order to obtain that in a whole neighborhood of $\mathcal{T}$ the symplectic form takes the form (3.26). The coordinate changes

we will use are not smooth (this would be impossible, since, due to our construction, the symplectic form Ω is not smooth), but they belong to a more general class that we are now going to define.

Definition 3.15. A map $\mathcal{T} : \mathcal{V}^{d_A/2} \to \mathcal{V}^{+\infty}$ (possibly only locally defined) is said to be *almost smooth* if there exist smoothing functions $q^j \in \mathcal{S}_l^i$ for some i, l, and $S_1^k \in \mathcal{S}_1^k$, for $k \geq 0$, such that the following representation formula holds

$$\mathcal{T}(\phi) = e^{q^j(\phi) J A_j}(\phi + S_1^k(\phi)) \ . \tag{3.27}$$

Remark 3.16. The range of the smoothing map S_1^k of equation (3.27) is not contained in $\mathcal{V}^\infty$, but in $\mathcal{H}^\infty$.

Remark 3.17. Almost smooth transformations form a group, furthermore for any j one has $\mathcal{P}_j(\mathcal{T}(\phi)) = \mathcal{P}_j(\phi) + R_2^1(\phi)$.

Proposition 3.18. *Let H be a Hamiltonian function defined on $\mathcal{H}^k$ for some k; assume that it is invariant under the symmetry group, namely that $H(u) = H(e^{q^j JA_j} u)$, and consider $H_r := i^* H$; let $\mathcal{T}$ be an almost smooth map with $q^j \in \mathcal{R}_2^1$ and $k = 1$ (vanishing index of the map S), then one has*

$$H_r(\mathcal{T}(\phi)) = H(\eta_{p_0 - N} + S_2^1 + \Pi_{p_0 - N}(\phi + S_1^1(\phi))) \ , \tag{3.28}$$

with suitable maps S_2^1 and S_1^1.

Proof. First, remark that for any choice of the scalar quantities q^j, and p_j,one has

$$\left\| (\Pi_p e^{q^j JA_j} - e^{q^j JA_j} \Pi_p)\phi \right\|_{\mathcal{H}^k} \leq C|q| \, \|\phi\|_{\mathcal{H}^{-l}} \ , \tag{3.29}$$

so that

$$\Pi_p e^{q^j JA_j} \phi = e^{q^j JA_j}(\Pi_p \phi + S_3^1) \tag{3.30}$$

Write $p' = p \circ \mathcal{T}$, $N' = \mathcal{P}(\mathcal{T}(\phi)) = N + R_2^1$, then one has $p' = p_0 - N' + R_2^1 \circ \mathcal{T} = p_0 - N' + R_2^1$. Consider now

$$\begin{aligned}
\eta_{p'} + \Pi_{p'}(\mathrm{e}^{q^j JA_j}(\phi + S_1^1)) &= \eta_{p_0 - N} + S_2^1 + \Pi_{p_0 - N}(\mathrm{e}^{q^j JA_j}(\phi + S_1^1)) + S_2^1 \\
&= \eta_{p_0 - N} + S_2^1 + \mathrm{e}^{q^j JA_j} \Pi_{p_0 - N}(\phi + S_1^1) + S_3^1 \\
&= \mathrm{e}^{q^j JA_j} \left(\mathrm{e}^{-q^j JA_j} \eta_{p_0 - N} + \mathrm{e}^{-q^j JA_j} S_2^1 + \Pi_{p_0 - N}(\phi + S_1^1)) \right) \\
&= \mathrm{e}^{q^j JA_j} \left(\eta_{p_0 - N} + S_2^1 + S_2^1 + \Pi_{p_0 - N}(\phi + S_1^1)) \right) \ .
\end{aligned}$$

Inserting in H and exploiting its invariance under the group action $\mathrm{e}^{q^j JA_j}$ one gets the result. ∎

Lemma 3.19. *Let $s^l \in \mathcal{R}_j^a$, $X \in \mathcal{S}_i^a$ $j \geq i \geq 1$, $a \geq 0$ be smoothing functions, and consider the equation*

$$\dot{\phi} = s^l(N, \phi) \Pi_{p_0} JA_l \phi + X \ . \tag{3.31}$$

Then for $|t|\leq 1$, the corresponding flow $\mathcal{F}_t$ exists in a sufficiently small neighborhood of the origin, and for any $|t|\leq 1$ it is an almost smooth transformation of the form

$$\mathcal{F}_t = \mathrm{e}^{q^l(N,\phi,t)JA_l}(\phi + tX(N,\phi,t) + S(N,\phi,t)) \ , \tag{3.32}$$

with $q^l(t)\in\mathcal{R}^a_j$ and $S(t)\in\mathcal{S}^a_{i+1}$. Furthermore one has

$$N(t) = N + t\langle X,\phi\rangle + S^a_{i+2} \ . \tag{3.33}$$

Proof. First rewrite (3.31) as

$$\dot{\phi} = s^l(N,\phi)JA_l\phi + X + S^a_{j+1} \ , \tag{3.34}$$

then we rewrite the equation (3.34) in a more convenient way, namely we add a separate equation for the evolution of N and then we use a variant of Duhamel principle in order to solve the system. Write

$$\dot{\phi} = Y(N,\phi) \ , \quad \dot{N} = Y_N(N,\phi) \ , \tag{3.35}$$

where

$$\begin{aligned} Y(N,\phi) &:= s^l(N,\phi)JA_l\phi + X + S^a_{i+1} \\ Y_{Nk} &= \langle\dot{\phi};A_k\phi\rangle = s_l\langle JA_l\phi;A_k\phi\rangle + \langle X+S^a_{i+1};\phi\rangle = \langle X+S^a_{i+1};\phi\rangle \ . \end{aligned}$$

In order to solve the system (3.35) we make the Ansatz $\phi = e^{q^jJA_j}\psi$ with q^j auxiliary variables. Compute $\dot{\phi}$, and impose $\dot{q}^l = s^l(N,\phi)$, thus we get the system

$$\dot{\psi} = \mathrm{e}^{-q^jJA_j}X(N,\mathrm{e}^{q^jJA_j}\psi) + S^a_{j+1}(N,\mathrm{e}^{q^jJA_j}\psi) \ , \tag{3.36}$$

$$\dot{q}^l = s^l(N,\mathrm{e}^{q^jJA_j}\psi) \tag{3.37}$$

$$\dot{N} = Y_N(N,\mathrm{e}^{q^jJA_j}\psi) \ , \tag{3.38}$$

which is equivalent to (3.34). Since the r.h.s. is smoothing, the system is well posed. To get the estimates ensuring the smoothing properties of the flow, just remark that we have

$$|\dot{q}| \leq C_l|N|^a\left\|\psi\right\|^2_{\mathcal{H}^{-l}} \ , \ |\dot{N}| \leq C_l|N|^a\left\|\psi\right\|^2_{\mathcal{H}^{-l}} \ , \ \left\|\dot{\psi}\right\|_{\mathcal{H}^k} \leq C_{kl}|N|^a\left\|\psi\right\|^i_{\mathcal{H}^{-l}} \ ,$$

then the standard theory of a priori estimates of differential equations gives the result. □

Remark 3.20. As we will see, in Darboux coordinates, the Hamiltonian vector field of a smoothing Hamiltonian has the structure (3.31), thus such Hamiltonians generate an almost smooth flow.

Theorem 3.21. *(Darboux theorem) There exists an almost smooth map*

$$\phi = \mathcal{F}(\phi') = e^{q^j J A_j}(\phi' + S^1_1(\phi')) \tag{3.39}$$

with $q^j \in \mathcal{R}^1_2$, such that $\mathcal{F}^*\Omega = \Omega_0$, i.e., in the coordinates ϕ' one has

$$\Omega(\Phi'_1; \Phi'_2) = \langle E\Phi'_1; \Phi'_2\rangle \ . \tag{3.40}$$

Correspondingly the Hamilton equations of a Hamilton function $H(\phi')$ have the form

$$\dot{\phi}' = \Pi_{p_0} J\nabla H(\phi') \tag{3.41}$$

The rest of the section is devoted to the proof Theorem 3.21.

We recall that in standard Darboux theorem the transformation introducing canonical coordinates is constructed as follows. Denote $\Omega_0 := \Omega_\phi\big|_{\phi=0}$, $\tilde{\Omega} := \Omega_0 - \Omega$ and $\Omega_t := \Omega + t\tilde{\Omega}$. Let α be a 1-form such that $\tilde{\Omega} = \mathrm{d}\alpha$ and let Y_t be such that $\Omega_t(Y_t, .) = -\alpha$. Let $\mathcal{F}_t$ be the evolution operator of Y_t (we will prove that it exists), then

$$\frac{d}{dt}\mathcal{F}^*_t\Omega_t = \mathcal{F}^*_t(\pounds_{Y_t}\Omega_t) + \mathcal{F}^*_t\frac{d}{dt}\Omega_t = \mathcal{F}^*_t(-d\alpha + \bar{\Omega}) = 0 \ , \tag{3.42}$$

so $\mathcal{F}^*_1\Omega_1 = \Omega$, and $\mathcal{F}^{-1}_1$ is the wanted change of variables. We follow such a scheme, by adding the explicit estimates showing that all the objects are well defined.

First we compute the expression of the symplectic form in the coordinates introduced by lemma 3.11. In order to simplify the computation we will first compute $\Omega^0 := i^*_0\omega$ with i_0 the map (3.16). It is also useful to compute a 1-form Θ^0 such that $\mathrm{d}\Theta^0 = \Omega^0$. Subsequently we compute $\Omega = i^*\omega$ and a potential 1-form for Ω by inserting the expression of $p = p(N, \phi)$.

Lemma 3.22. *Define the 1-form Θ^0 by*

$$\Theta^0(P, \Phi) = \frac{1}{2}\langle E\Pi_p\phi; \frac{\partial \Pi_p}{\partial p_j}\phi\rangle P_j + \langle E\Pi_p\phi; \Phi\rangle \tag{3.43}$$

(by this notation we mean that the r.h.s. gives the action of the form Θ^0 at the point (p, ϕ) on a vector (P, Φ)), then one has $\mathrm{d}\Theta^0 = \Omega^0 \equiv i^*_0\omega$, and therefore

$$\begin{aligned} \Omega^0\left((P_1, \Phi_1); (P_2, \Phi_2)\right) = &\left\langle E\frac{\partial \Pi_p}{\partial p_j}\phi, \frac{\partial \Pi_p}{\partial p_i}\phi\right\rangle P_{1j}P_{2i} + \left\langle E\Pi_p\frac{\partial \Pi_p}{\partial p_j}\phi; \Phi_2\right\rangle P_{1j} \\ &- \left\langle E\Pi_p\frac{\partial \Pi_p}{\partial p_j}\phi; \Phi_1\right\rangle P_{2j} + \langle E\Pi_p\Phi_1; \Phi_2\rangle \ . \end{aligned} \tag{3.44}$$

Proof. We compute $i^*_0\theta$, where $\theta = \langle Eu; .\rangle/2$ is such that $\omega = \mathrm{d}\theta$. By writing $u = i^*_0(p, \phi)$, one has

$$\frac{\partial u}{\partial p_j} = \frac{\partial \eta_p}{\partial p_j} + \frac{\partial \Pi_p}{\partial p_j}\phi \ , \quad (\mathrm{d}_\phi i_0)\Phi = \Pi_p\Phi \tag{3.45}$$

so, taking $\theta = \frac{1}{2}\langle Eu; . \rangle$, one has

$$(i_0^*\theta)(P,\Phi) = \frac{1}{2}\left\langle Eu; \frac{\partial u}{\partial p_j}\right\rangle P_j + \frac{1}{2}\left\langle Eu; \mathrm{d}_\phi i_0 \Phi\right\rangle \ .$$

We compute the first term, which coincides with

$$\begin{aligned} 2\theta\left(\frac{\partial u}{\partial p_j}\right) &= \langle E(\eta_p + \Pi_p\phi); \frac{\partial \eta_p}{\partial p_j} + \frac{\partial \Pi_p}{\partial p_j}\phi\rangle \\ &= \langle E\eta_p; \frac{\partial \eta_p}{\partial p_j}\rangle + \langle E\eta_p; \frac{\partial \Pi_p}{\partial p_j}\phi\rangle + \langle E\Pi_p\phi; \frac{\partial \eta_p}{\partial p_j}\rangle + \langle E\Pi_p\phi; \frac{\partial \Pi_p}{\partial p_j}\phi\rangle \ . \end{aligned} \tag{3.46}$$

Now, the third term vanishes due to the definition of Π_p. Concerning the first term, there exists a function f_0 such that $\frac{\partial f_0}{\partial p_j} = \langle E\eta_p; \frac{\partial \eta_p}{\partial p_j}\rangle$, indeed, from the isotropy property (B3) one has

$$\frac{\partial}{\partial p_j}\langle E\eta_p; \frac{\partial \eta_p}{\partial p_i}\rangle = \frac{\partial}{\partial p_i}\langle E\eta_p; \frac{\partial \eta_p}{\partial p_j}\rangle \ .$$

Finally, defining $f_1(p,\phi) = \langle E\eta_p; \Pi_p\phi\rangle$, the second term of (3.46) turns out to be given by $\frac{\partial f_1}{\partial p_j}$, so we have

$$2\theta\left(\frac{\partial u}{\partial p_j}\right) = \langle E\Pi_p\phi; \frac{\partial \Pi_p}{\partial p_j}\phi\rangle + \frac{\partial(f_0+f_1)}{\partial p_j} \ .$$

We compute now $\langle Eu; (\mathrm{d}_\phi i_0)\Phi\rangle$. We have

$$2\langle Eu; (\mathrm{d}_\phi i_0)\Phi\rangle = \langle E(\eta_p + \Pi_p\phi); \Pi_p\Phi\rangle = \langle E\Pi_p\phi; \Pi_p\Phi\rangle + (\mathrm{d}_\phi f_1)\Phi \ ,$$

from which $i_0^*\theta = \Theta^0 + \mathrm{d}(f_0 + f_1)$, and therefore $\Omega^0 = \mathrm{d}\Theta^0$.

We compute now explicitly $\mathrm{d}\Theta^0$. Denote $\Theta^0 = \Theta^{0j}\mathrm{d}p_j + \langle\tilde{\Theta}^0_\phi; .\rangle$, then the computation of $\frac{\partial\Theta^{0i}}{\partial p_j} - \frac{\partial\Theta^{0j}}{\partial p_i}$ is trivial and gives the term proportional to $P_{1j}P_{2i}$ in (3.44). Also the computation of the term containing Φ_1, Φ_2 is trivial and is omitted. We come to the P, Φ terms. When applied to a vector Φ it is given by

$$\langle\frac{\partial\Theta^0_\phi}{\partial p_j}; \Phi\rangle - (\mathrm{d}_\phi\Theta^0_j)\Phi \tag{3.47}$$

$$= \frac{1}{2}\left(\left\langle E\frac{\partial \Pi_p}{\partial p_j}\phi; \Phi\right\rangle - \left\langle E\Pi_p\Phi; \frac{\partial \Pi_p}{\partial p_j}\phi\right\rangle - \left\langle E\Pi_p\phi; \frac{\partial \Pi_p}{\partial p_j}\Phi\right\rangle\right) \ , \tag{3.48}$$

which is the scalar product of Φ with a half of the vector

$$\begin{aligned} E\frac{\partial \Pi_p}{\partial p_j}\phi + \Pi_p^* E\frac{\partial \Pi_p}{\partial p_j}\phi - \frac{\partial \Pi_p^*}{\partial p_j}E\Pi_p\phi &= E\left(\frac{\partial \Pi_p}{\partial p_j} + \Pi_p\frac{\partial \Pi_p}{\partial p_j} - \frac{\partial \Pi_p}{\partial p_j}\Pi_p\right)\phi \\ &= E\left(\Pi_p\frac{\partial \Pi_p}{\partial p_j} + \frac{\partial \Pi_p}{\partial p_j}\Pi_p + \Pi_p\frac{\partial \Pi_p}{\partial p_j} - \frac{\partial \Pi_p}{\partial p_j}\Pi_p\right)\phi = 2E\Pi_p\frac{\partial \Pi_p}{\partial p_j}\phi \ , \end{aligned}$$

which immediately gives the thesis. ∎

Lemma 3.23. *In the coordinates of lemma 3.11 the symplectic form $\Omega = i^*\omega$ takes the form $\Omega(\Phi_1, \Phi_2) = \langle O\Phi_1; \Phi_2\rangle$ with O given by*

$$O\Phi = a^{ij}\langle \nabla p_i; \Phi\rangle \nabla p_j + \langle \nabla p_j; \Phi\rangle \Pi_{p_0}^* E\Pi_p \frac{\partial \Pi_p}{\partial p_j}\phi \tag{3.49}$$

$$- \left\langle E\Pi_p \frac{\partial \Pi_p}{\partial p_j}\phi; \Phi \right\rangle \nabla p_j + \Pi_{p_0}^* E\Pi_p \Phi \tag{3.50}$$

where

$$a^{ij} := \langle E\frac{\partial \Pi_p}{\partial p_i}\phi; \frac{\partial \Pi_p}{\partial p_j}\phi\rangle \ . \tag{3.51}$$

Moreover one has $\Omega = \mathrm{d}\Theta$ with

$$\Theta(\Phi) = \frac{1}{2}\langle E\Pi_p\phi; \Phi\rangle + \frac{1}{2}\left\langle E\Pi_p\phi; \frac{\partial \Pi_p}{\partial p_j}\phi \right\rangle \langle \nabla p_j; \Phi\rangle \tag{3.52}$$

Proof. The expression of Ω and Θ are obtained by taking (3.43) and (3.44) and inserting the expression of $p = p(\phi) \equiv p(\mathcal{P}(\phi), \phi)$ and substituting $P_{1,2j} = \langle \nabla p_j; \Phi_{1,2}\rangle$, thus the thesis follows from a simple computation. □

Remark 3.24. One can define $\Omega_t = \langle O_t.; .\rangle$ and $\alpha = \langle V; .\rangle$ with

$$O_t = E + (1-t)\left[\Pi_{p_0}^* E\Pi_p - E + a_{ij}\langle \nabla p_i; .\rangle \nabla p_j \right. \tag{3.53}$$

$$\left. + \langle \nabla p_i; .\rangle \Pi_{p_0}^* E\Pi_p \frac{\partial \Pi_p}{\partial p_j}\phi - \left\langle E\Pi_p \frac{\partial \Pi_p}{\partial p_j}; . \right\rangle \nabla p_j \right]$$

$$V = \frac{1}{2}\left(\Pi_{p_0}^* E\Pi_p\phi - E\phi + \left\langle E\Pi_p\phi; \frac{\partial \Pi_p}{\partial p_j}\phi \right\rangle \nabla p_j \right) \tag{3.54}$$

In order to find the normalizing vector field Y_t, we have to solve the equation

$$O_t Y = -V \tag{3.55}$$

where we omitted the index t from Y. We start now the discussion of such an equation.

First we have the lemma:

Lemma 3.25. *Define $D_t := E + (1-t)(\Pi_{p_0}^* E\Pi_p - E)$; this is a skewsymmetric operator. Provided $|p - p_0|$ is small enough $\exists S_t$ fulfilling $S_t(\mathcal{V}^{-\infty})^* \subset \mathcal{V}^\infty$, s.t. $D_t^{-1} = J + S_t$ and*

$$\|S_t\Phi\|_{\mathcal{H}^k} \le C_{k,l}\,|p - p_0|\,\|\Phi\|_{-l} \ . \tag{3.56}$$

Proof. We have $D_t = E + (1-t)\tilde{D}$, where $\tilde{D} := \Pi_{p_0}^* E\Pi_p - E$, which is smoothing and fulfills an inequality equal to (3.56). Then $D_t = E(\mathbb{1} + (1-t)J\tilde{D})$, and by Neumann formula one gets

$$D_t^{-1} = J + \sum_{k\ge 1}(-1)^k((1-t)J\tilde{D})^k J$$

and the thesis. □

Lemma 3.26. *The solution of equation* (3.55) *has the form*

$$Y(p,\phi) = s^l(p,\phi,t)\Pi_{p_0} J A_l \phi + S_1^1(t) \tag{3.57}$$

where $s^j(t) \in \mathcal{R}_2^1$, $S_1^1(t) \in \mathcal{S}_1^1$ *and* $S_1^1 \mathcal{V}^{-\infty} \subset \mathcal{V}^\infty$ *are smoothly dependent on* $t \in [0,1]$.

Proof. First write explicitly (3.55) introducing, for short, the notations

$$b_i := \langle \nabla p_i, Y \rangle \ , \quad W^i = \Pi_{p_0}^* E \Pi_p \frac{\partial \Pi_p}{\partial p_i} \phi \ ,$$

so that it takes the form

$$\begin{aligned} D_t Y + (1-t)\left(a^{ij} b_i \nabla p_j + b_i W^i - \left\langle W^j, Y \right\rangle \nabla p_j\right) \\ = -\frac{1}{2}[E\Pi_p - E]\phi + \frac{1}{2}\langle \phi; W^j \rangle \nabla p_j \ . \end{aligned} \tag{3.58}$$

Applying D_t^{-1} and reordering the formula one gets

$$\begin{aligned} Y = -(1-t)\left(a^{ij} b_i D_t^{-1} \nabla p_j + b_i D_t^{-1} W^i - \langle W_j; Y \rangle D_t^{-1} \nabla p_j\right) \\ + \frac{1}{2}\langle \phi; W^j \rangle D_t^{-1} \nabla p_j - \frac{1}{2} D_t^{-1} [E\Pi_p - E]\phi \ . \end{aligned} \tag{3.59}$$

Denote $\gamma_{ij} := \left\langle \nabla p_i; D_t^{-1} \nabla p_j \right\rangle$, $\beta_i^j := \langle \nabla p_i; D_t^{-1} W^j \rangle$ and remark that γ_{ij} is smoothing, since it is given by

$$\begin{aligned} \langle \nabla p_i; J \nabla p_j \rangle + \langle \nabla p_i; S_t \nabla p_j \rangle = M_i^l \langle A_l \phi; J A_k \phi \rangle M_j^k + \langle \nabla p_i; S_t \nabla p_j \rangle \\ = \langle \nabla p_i; S_t \nabla p_j \rangle \ . \end{aligned}$$

Also β_i^j is clearly smoothing. Now one has

$$\begin{aligned} b_i &= -(1-t)\Big(\gamma_{ij} a^{ik} b_k + \beta_i^j b_j - \langle W^j; Y \rangle \gamma_{ij}\Big) \\ &+ \gamma_{ij} \frac{1}{2} \langle \phi; W^j \rangle - \frac{1}{2} \langle \nabla p_i; D_t^{-1} \left[E\Pi_p - E \right] \phi \rangle \\ \langle W^l; Y \rangle &= -(1-t)\left(a^{ij} b_i \beta_j^l + b_i \langle W^l; D_t^{-1} W^i \rangle - \langle W^j; Y \rangle \beta_j^l\right) \\ &+ \frac{1}{2} \langle \phi; W^j \rangle \beta_j^l - \frac{1}{2} \langle W^l; D_t^{-1} [E\Pi_p - E]\phi \rangle \ , \end{aligned}$$

which is a linear system for b_i and $\langle W^l; Y \rangle$. Solving it and analyzing the solutions one gets

$$b_i = -\frac{1}{2} \langle \nabla p_i; D_t^{-1} \left[E\Pi_p - E \right] \phi \rangle + \text{h.o.t.} \ , \tag{3.60}$$

$$\langle W^l; Y \rangle = -\frac{1}{2} \langle W^l; D_t^{-1} [E\Pi_p - E]\phi \rangle + \text{h.o.t.} \ , \tag{3.61}$$

where h.o.t are also regularizing. Substituting in (3.59) one gets a formula for Y. Then one has that such an Y actually fulfills (3.58), and is thus the wanted solution of (3.55). From these formulae one immediately has

$$|b_i| \le C\,|p-p_0|\,\|\phi\|^2_{\mathcal{H}^{-k}} \le CN\,\|\phi\|^2_{\mathcal{H}^{-k}} \tag{3.62}$$

$$|\langle W^l;Y\rangle| \le C\,|p-p_0|^2\,\|\phi\|^2_{\mathcal{H}^{-k}} \le CN^2\,\|\phi\|^2_{\mathcal{H}^{-k}} \tag{3.63}$$

To get the formula (3.57) and the corresponding estimates, define the function s^l to be the coefficient of $D_t^{-1}\nabla p_l$ in (3.59), and remark that $D_t^{-1}\nabla p_l = \Pi_{p_0} J A_l \phi$+smoothing terms. Then it is easy to conclude the proof. □

Theorem 3.21 is an immediate consequence of Lemma 3.19.

4 The Hamiltonian in Darboux coordinates

Concerning the smoothness and the structure of the nonlinear part of the Hamiltonian we make the following assumption

(P2) The map

$$\mathcal{H}^\infty \ni \eta \mapsto \mathrm{d}H_P(\eta) \tag{4.1}$$

is C^∞ as a map taking values in the space of linear functionals on $\mathcal{H}^{-\infty}$. The same is true for the map

$$\mathcal{H}^\infty \times \mathcal{H}^\infty \ni (\eta,\Phi) \mapsto \mathrm{d}^2H_P(\eta)(\Phi;.)\ . \tag{4.2}$$

4.1 The Hamiltonian in Darboux coordinates

We introduce the coordinates of the Darboux theorem 3.21. To this end we exploit proposition 3.18 from which one gets:

Proposition 4.1. *In the Darboux coordinates introduced by theorem 3.21 the Hamiltonian $H_r \circ \mathcal{F}$ has the form*

$$H_r(\mathcal{F}(\phi)) = H_L + H_N\ , \tag{4.3}$$

$$H_L := \mathcal{P}_0(\phi) + \frac{1}{2}\mathrm{d}^2H_P(\eta_{p_0-N})(\phi,\phi) - \lambda^j(p_0)\mathcal{P}_j + D(N) + (R^1_2)_{lin} \tag{4.4}$$

$$H_N := R^1_3 + H^3_P(\eta_{p_0-N},\phi+S^1_1)$$

where $(R^1_2)_{lin}$ is a smoothing quadratic polynomial in ϕ and

$$D(N) := H(\eta_{p_0-N}) - [H(\eta_{p_0}) - \frac{\partial H}{\partial p_j}(\eta_{p_0})N_j]\ , \tag{4.5}$$

$$H^3_P(\eta,\phi) := H_P(\eta+\phi) - [H_P(\eta) + \mathrm{d}H_P(\eta)\phi + \frac{1}{2}\mathrm{d}^2H_P(\eta)(\phi,\phi)]\ . \tag{4.6}$$

Proof. Exploiting proposition 3.18 one has to study

$$\begin{aligned}
&\mathcal{P}_0(\eta_{p_0-N} + S_2^1 + \Pi_{p_0-N}(\phi + S_1^1)) = \mathcal{P}_0(\eta_{p_0-N} + \Pi_{p_0-N}(\phi + S_1^1)) + R_2^1 \\
&= \mathcal{P}_0(\eta_{p_0-N}) + \left\langle A_0\eta_{p_0-N}; \Pi_{p_0-N}(\phi + S_1^1)\right\rangle + \mathcal{P}_0(\Pi_{p_0-N}(\phi + S_1^1)) + R_2^1 \ ,
\end{aligned}$$

and

$$\begin{aligned}
&H_P(\eta_{p_0-N} + S_2^1 + \Pi_{p_0-N}(\phi + S_1^1)) \\
&= H_P(\eta_{p_0-N}) + \mathrm{d}H_P(\eta_{p_0-N})\left(S_2^1 + \Pi_{p_0-N}(\phi + S_1^1)\right) \\
&\quad + \frac{1}{2}\mathrm{d}^2 H_P(\eta_{p_0-N})\left(S_2^1 + \Pi_{p_0-N}(\phi + S_1^1)\right)^{\otimes 2} \\
&\quad + H_P^3(\eta_{p_0-N}; S_2^1 + \Pi_{p_0-N}(\phi + S_1^1)) \ .
\end{aligned}$$

Consider first the terms linear in ϕ: they are given by

$$\begin{aligned}
&\left\langle A_0\eta_{p_0-N}; \Pi_{p_0-N}(\phi + S_1^1)\right\rangle + \mathrm{d}H_P(\eta_{p_0-N})\left(\Pi_{p_0-N}(\phi + S_1^1)\right) \\
&= \lambda^j(p_0 - N)\left\langle A_j\eta_{p_0-N}; \Pi_{p_0-N}(\phi + S_1^1)\right\rangle = 0
\end{aligned}$$

where we exploited eq. (3.3). Thus we have

$$\begin{aligned}
H_r(\mathcal{F}(\phi)) &= H(\eta_{p_0-N}) + \mathcal{P}_0(\phi + S_1^1) + \frac{1}{2}\mathrm{d}^2 H_P(\eta_{p_0-N})(\phi + S_1^1, \phi + S_1^1) \\
&\quad + R_2^1 + H_P^3(\eta_{p_0-N}, \phi + S_1^1) \\
&= H(\eta_{p_0-N}) + \mathcal{P}_0(\phi) + R_2^1 + \frac{1}{2}\mathrm{d}^2 H_P(\eta_{p_0-N})(\phi, \phi) \\
&\quad + R_2^1 + H_P^3(\eta_{p_0-N}, \phi + S_1^1) \ .
\end{aligned}$$

Finally we have to rewrite in a suitable form the function $H(\eta_{p_0-N})$. To this end remark that, due to equation (3.3), one has

$$\frac{\partial f}{\partial p_j}(p_0) = \mathrm{d}H(\eta_{p_0})\frac{\partial \eta_{p_0}}{\partial p_j} = \lambda^k \langle A_k\eta_{p_0}; \frac{\partial \eta_{p_0}}{\partial p_j}\rangle = \lambda^k \delta_k^j = \lambda^j \ , \tag{4.7}$$

from which the thesis immediately follows. □

Remark 4.2. Define $X_P := J\nabla H_P$, and, for fixed $\eta \in \mathcal{H}^\infty$,

$$X_P^2(\eta; \phi) := X_P(\eta + \phi) - [X_P(\eta) + \mathrm{d}X_P(\eta)\phi] \ , \tag{4.8}$$

then one has $J\nabla H_P^3(\eta; \phi) = X_P^2(\eta, \phi)$. This can be seen by writing the definition of Hamiltonian vector field.

Remark 4.3. The Hamilton vector field of $H_r \circ \mathcal{F}$**, which from now on will be simply denoted by** H is given by

$$\begin{aligned}
\dot{\phi} = \Pi_{p_0} J[A_0\phi + V_N\phi + S_1^1(N, \phi) + X_P^2(\eta_{P_0-N}, \phi) - \lambda^j(p_0)A_j\phi] \\
+ w^j(N, \phi)\Pi_{p_0} JA_j\phi \ ,
\end{aligned} \tag{4.9}$$

where

$$w^j := \lambda^j(p_0) - \lambda^j(p_0 - N) + \frac{1}{2}\langle \frac{\partial V_N}{\partial N_j}\phi; \phi\rangle + S_2^0 + \frac{\partial H_N}{\partial N_j} \tag{4.10}$$

and V_N is the operator such that

$$d^2 H_P(\eta_{p_0 - N})(\phi, \phi) = \frac{1}{2}\langle V_N \phi; \phi\rangle \ , \tag{4.11}$$

so that $V_N\phi = \mathrm{d}X_P(\eta_{p_0-N})\phi$.

4.2 Adapted coordinates

Consider the quadratic part of the original Hamiltonian at η_{p_0}, namely

$$H_{L0}(u) := \mathcal{P}_0(u) + \frac{1}{2}\mathrm{d}^2 H_P(\eta_{p_0})(u, u) - \lambda^j(p_0)\mathcal{P}_j(u) \tag{4.12}$$

Denote $B := A_0 + V_0 - \lambda^j(p_0)A_j$, $L_0 := JB$, so that $H_{L0}(u) = \langle u; Bu\rangle/2 = \langle EL_0 u; u\rangle/2$. Making the Ansatz $u = \mathrm{e}^{t\lambda^j JA_j}(\eta_{p_0} + \chi)$ and linearizing in χ the Hamilton equations of (3.1), one gets that χ satisfies $\dot{\chi} = L_0\chi$.

Lemma 4.4. *The generalized kernel of L_0^* contains the vectors*

$$A_j\eta_{p_0} \ , \quad E\frac{\partial \eta_{p_0}}{\partial \lambda^j} \ . \tag{4.13}$$

Proof. First, one immediately sees that $JA_j\eta_{p_0} \in Ker(L_0)$, then exploiting the equation (3.3) for the ground state one sees that $B\frac{\partial \eta_p}{\partial \lambda^j} = A_j\eta_p$, which implies $[L_0^*]^2 E\frac{\partial \eta_{p_0}}{\partial \lambda^j} = [(JB)^*]^2 E\frac{\partial \eta_{p_0}}{\partial \lambda^j} = 0$, from which one immediately sees that the generalized kernel of $(L_0)^*$ contains the vectors (4.13). □

We assume

(L1) The genralized Kernel of L_0^* is $2n$-dimensional. Furthermore $L_0\big|_{\mathcal{V}^j}$ is an isomorphism between $\mathcal{V}^j$ and $\mathcal{V}^{j-d_0}$.

(L2) $\langle B\phi, \phi\rangle > 0$, $\forall \phi \neq 0$, $\phi \in \mathcal{V}^{d_0/2}$.

(L3) the essential spectrum of $L_0\big|_{\mathcal{V}^0}$ is $\bigcup_{\pm} \pm\mathrm{i}[\Omega, +\infty)$. The rest of the spectrum consists of purely imaginary eigenvalues $\pm\mathrm{i}\omega_j$, that we order as follows $0 < \omega_1 \leq \omega_2 \leq ... \leq \omega_K$. Furthermore the corresponding eigenfunctions $v_{j\pm}$ are smooth, namely $v_{j\pm} \in (\mathcal{V}^\infty)^{\otimes\mathbb{C}}$.

In order to perform the dispersive estimates we will also have to avoid boundary resonances. Let r_t be the smallest integer such that $r_t\omega_1 \geq \Omega$. We assume that

(L4) One has $\omega \cdot k \neq \Omega$, $\forall k \in \mathbb{Z}^K$ s.t. $|k| \leq 2r_t$.

We normalize the eigenfunctions in such a way that

$$\langle Ev_{j\pm}, v_{k\pm}\rangle = 0\ ,\ \forall j,k\ ,\quad \langle Ev_{j\pm}, v_{k\mp}\rangle = \mathrm{i}\delta_{jk}\ ,\quad \overline{v_{j\pm}} = v_{j\mp}\ , \tag{4.14}$$

which is always possible since (4.14) are the standard "symplectic orthogonality" relations of the eigenfunctions of the operator L_0 (which is skew with respect to the symplectic form).

We now introduce coordinates (ξ_j, ϕ_c) by

$$\phi = \sum_{j=1}^{K}(\xi_j v_{j+} + \bar{\xi}_j v_{j-}) + \phi_c\ , \tag{4.15}$$

where ϕ_c is such that $\langle Ev_{j\pm};\phi_c\rangle = 0$ $\forall j$'s. Explicitely one has

$$\xi_j = \mathrm{i}\langle Ev_{j-}, \phi\rangle\ ,\quad \bar{\xi}_j = -\mathrm{i}\langle Ev_{j+}, \phi\rangle\ , \tag{4.16}$$

$$\phi_c := P_c\phi := \phi - \sum_{j,\pm} \pm\mathrm{i}\langle Ev_{j\mp};\phi\rangle v_{j\pm}\ . \tag{4.17}$$

In these coordinates the phase space becomes

$$(\xi, \phi_c) \in \mathbb{C}^n \oplus \mathcal{W}^j\ ,\quad \mathcal{W}^j := P_c\mathcal{V}^j\ . \tag{4.18}$$

As usual it is often usefull to consider the variables $\bar{\xi}_j$ as independent from the ξ_j's. Often we will also denote $\phi_d := \sum_{j=1}^{K}(\xi_j v_{j+} + \bar{\xi}_j v_{j-})$. In these coordinates the Hamiltonian vector field of a Hamiltonian function H takes the form

$$\dot{\xi}_j = -\mathrm{i}\frac{\partial H}{\partial \bar{\xi}_j}\ ,\quad \dot{\phi}_c = J\nabla_{\phi_c}H\ ,$$

and the main term of the quadratic part of the Hamiltonain (3.1) takes the form

$$H_{L0} = \sum_l \omega_l|\xi_l|^2 + \frac{1}{2}\langle EL_c\phi_c;\phi_c\rangle\ , \tag{4.19}$$

where $L_c := P_cL_0P_c$. Concerning the momenta $\mathcal{P}_j$ one has

$$\mathcal{P}_j(\xi, f) = \frac{1}{2}\langle\phi_c;\mathcal{A}_j\phi_c\rangle + P_1^j(\phi_c, \xi) + P_2^j(\xi)\ , \tag{4.20}$$

where

$$P_1^j(\phi_c, \xi) := \sum_{|\alpha|+|\beta|=1}\langle\phi_c; E\Phi^j_{\alpha\beta}\rangle\xi^\alpha\bar{\xi}^\beta\ ,\quad P_2^j(\xi) := \sum_{|\alpha|+|\beta|=2} A^j_{\alpha\beta}\xi^\alpha\bar{\xi}^\beta\ , \tag{4.21}$$

and $\Phi^j_{\alpha\beta} \in (\mathcal{W}^\infty)^{\otimes\mathbb{C}}$, $A^j_{\alpha\beta}$ are suitable functions and complex numbers, while $\mathcal{A}_j := P_cA_jP_c$. By a small abuse of notation, in the following we will always denote in the same way $\mathcal{W}^j$ and its complexification. Denote by M_j the function

$$M_j := \frac{1}{2}\langle\phi_c;\mathcal{A}_j\phi_c\rangle\ . \tag{4.22}$$

Since P_1 and P_2 are smoothing, in the following the quantities M_j will play the role that in the previous sections was plaid by the quantities N_j.

In the following we will substitute the classes $\mathcal{R}_l^k$ by similar classes in which the functions N are substituted by the functions M and similarly for the classes $\mathcal{S}_l^k$.

In order to make the translation we remark that, if $F \in \mathcal{R}_2^1$ **old classes**, then we have

$$F = R_2^1 + R_4^0 \quad \text{new classes} \tag{4.23}$$

as it immediately follows from (4.20). Similarly one has

$$\text{old} \quad S_1^1 \text{``}=\text{''}\ S_1^1 + S_2^0 \ , \quad \text{old} \quad R_2^1 \text{``}=\text{''}\ R_2^1 + R_3^0 \ .$$

Remark 4.5. From the definition of the operators $\mathcal{A}_j$ we have that they do not fulfill assumption (S1) and (S2), instead they fulfill

$$[\mathcal{A}_j ; \mathcal{A}_k] = S_{jk} \ , \quad [\mathcal{A}_k ; J] = S_k \tag{4.24}$$

with S_{jk} and S_k smoothing operators.

Lemma 4.6. *One has* $\mathrm{e}^{tJ\mathcal{A}_j}\phi_c = e^{tJA_j}\phi_c + S(t)\phi_c$, *where S is a smoothing family of operators which fulfills* $\left\|S(t)\phi_c\right\|_k \preceq |t| \left\|\phi_c\right\|_{-l}$.

Proof. Write explicitely the equation $\dot{\phi} = JA_j\phi$ (which defines e^{tJA}):

$$\dot{\phi}_c = J\mathcal{A}_j\phi_c + \sum_{|\alpha+\beta|=1} \xi^\alpha \bar{\xi}^\beta \Phi^j_{\alpha\beta} \ , \tag{4.25}$$

$$\dot{\xi}_k = -\mathrm{i}\left(\sum_{|\alpha+\beta|=2} \frac{\beta_k}{\bar{\xi}_k} A^j_{\alpha\beta}\xi^\alpha\bar{\xi}^\beta + \sum_{|\alpha+\beta|=1} \frac{\beta_k}{\bar{\xi}_k} \left\langle E\Phi^j_{\alpha\beta} ; \phi_c \right\rangle \xi^\alpha\bar{\xi}^\beta \right) \ . \tag{4.26}$$

Using Duhamel formula (with $J\mathcal{A}_j$ as principal part), one gets the thesis. ☐

Remark 4.7. If $\chi(M, \xi, \phi_c)$ is a smoothing Hamiltonian then, by lemma 3.19, its Hamiltonian vector field generates an almost smooth map.

With the new notations and classes one has that the Hamiltonian of the system takes the form

$$H = H_L + H_N \ , \quad H_L = H_{L0} + H_{L1} + D(M) \ , \tag{4.27}$$

$$H_{L0} := \mathcal{P}_0(\phi) + \frac{1}{2}\langle V_0\phi ; \phi\rangle + \lambda^j(p_0)\mathcal{P}_j(\phi) = \sum_l \omega_l |\xi_l|^2 + \frac{1}{2}\langle EL_c\phi_c ; \phi_c\rangle \ , \tag{4.28}$$

$$H_{L1} = \frac{1}{2}\langle (V_M - V_0)\phi ; \phi\rangle + (\mathcal{R}_2^1)_{lin} \tag{4.29}$$

$$H_N := R_3^1 + R_4^0 + H_P^3(\eta_{p_0 - M} ; \phi + S_1^1 + S_2^0) \ , \tag{4.30}$$

where V_M is the operator V_N evaluated at $N = M$.

Of course, this is also true for $H \circ \mathcal{T}$ with any $\mathcal{T}$ almost smooth of the form (3.39).

In the following we will denote by X_N the vector field of H_N computed at constant M, i.e. as if M were independent of ϕ_c.

5 Normal Form

First we define what we mean by normal form.

Definition 5.1. A function $Z(M, \xi, \phi_c)$, $Z \in C^\infty(\mathbb{R}^n \times \mathcal{V}^\infty)$, will be said to be in normal form at order r, if the following holds

$$\{\omega \cdot (\mu - \nu) \neq 0 \;\&\; |\mu| + |\nu| \leq r\} \Longrightarrow \frac{\partial^{|\mu|+|\nu|} Z}{\partial \xi^\mu \partial \bar{\xi}^\nu}(M, 0) = 0 \tag{5.1}$$

$$\{|\omega \cdot (\mu - \nu)| < \Omega \;\&\; |\mu| + |\nu| \leq r - 1\} \Longrightarrow \mathrm{d}_{\phi_c} \frac{\partial^{|\mu|+|\nu|} Z}{\partial \xi^\mu \partial \bar{\xi}^\nu}(M, 0) = 0 \tag{5.2}$$

The derivatives with respect to ϕ_c have to be computed at constant M, i.e. as if M were independent of such quantities.

Theorem 5.2. *For any $r \geq 2$ there exists an almost smooth canonical transformation $\mathcal{T}_r(\phi) = \mathrm{e}^{q^l J A_l}(\phi + S(\phi))$, with $q^l \in \mathcal{R}_2^0$, and $S \in \mathcal{S}_1^1 \cup \mathcal{S}_2^0$, such that $H \circ \mathcal{T}_r$ is in normal form at order r.*

The rest of the section is devoted to the proof of theorem 5.2.

In order to put the system in normal form we will use the method of Lie transform that we now recall. Having fixed $r \geq 2$, consider a function χ of the form

$$\chi := \sum_{\substack{|\mu+\nu|=r \\ \omega\cdot(\nu-\mu)\neq 0}} \chi_{\mu\nu} \xi^\mu \bar{\xi}^\nu + \sum_{\substack{|\mu|+|\nu|=r-1 \\ |\omega\cdot(\nu-\mu)|<\Omega}} \xi^\mu \bar{\xi}^\nu \left\langle E\Phi_{\mu\nu}; \phi_c \right\rangle \tag{5.3}$$

where $\Phi_{\mu\nu} \in \mathcal{W}^\infty$. It is useful to identify the space of the functions of the form (5.3) with the linear space of its coefficients $(\chi_{\mu\nu}; \Phi_{\mu\nu})$. When endowed by the norm

$$\|\chi\|_k := \sup_{\mu\nu} |\chi_{\mu\nu}| + \sup_{\mu\nu} \|\Phi_{\mu\nu}\|_{\mathcal{W}^k} \;, \tag{5.4}$$

it will be denoted by Gen_k; we will mainly consider functions $\mathbb{R}^n \ni M \mapsto \chi(M) \in \mathrm{Gen}_k$, of class C^s, namely $\chi \in C^s(\mathbb{R}^n, \mathrm{Gen}_k)$.

According to remark 4.7 the flow ϕ^t of X_χ exists up to time 1 in a neighborhood of the origin, then we will denote $\mathcal{T} := \phi^1 \equiv \phi^t\big|_{t=1}$. Such a transformation will be called the *Lie transform* generated by χ.

We define now the nonresonant projector Π_{nr} acting on homogeneous polynomials; it restricts the sum to nonresonant values of the indexes. So let $F = F(M, \phi)$ be a homogeneous polynomial of degree r in ϕ. Consider first $F_1(M, \phi) := F(M, \phi_d) + \mathrm{d}F(M, \phi_d)\phi_c$, where d is the differential at fixed M.

$$F_1(M, \xi, \phi_c) = \sum_{|\mu+\nu|=r} F_{\mu\nu}(M) \xi^\mu \bar{\xi}^\nu + \sum_{|\mu|+|\nu|=r-1} \xi^\mu \bar{\xi}^\nu \left\langle E\Phi^F_{\mu\nu}(M); \phi_c \right\rangle \;, \tag{5.5}$$

in general $\Phi^F_{\mu\nu} \in \mathcal{W}^{-j}$, but we will see that in the cases we will meet we always have $\Phi^F_{\mu\nu} \in \mathcal{W}^{\infty}$. We define

$$\Pi_{nr} F(M,\phi) := \sum_{\substack{|\mu+\nu|=r \\ \omega\cdot(\nu-\mu)\neq 0}} F_{\mu\nu}(M)\xi^{\mu}\bar{\xi}^{\nu} + \sum_{\substack{|\mu|+|\nu|=r-1 \\ |\omega\cdot(\nu-\mu)|<\Omega}} \xi^{\mu}\bar{\xi}^{\nu}\left\langle E\Phi^F_{\mu\nu}(M);\phi_c\right\rangle \ . \tag{5.6}$$

Finally, given $F \in C^{\infty}(\mathcal{H}^{\infty})$, we define the projector Π^r_{nr} which by definition produces the nonresonant part of the homogeneous Taylor polynomial of degree r of F.

In order to prove theorem 5.2 we proceed iteratively: we assume the system to be in normal form at order $r-1$ and we normalize it at order r. In order to perform the $r-th$ step we look for a function $\chi_r(M,\phi) \in C^{\infty}(\mathbb{R}^n, \mathrm{Gen}_k)$ such that the corresponding Lie transform $\mathcal{T}_r$ is the wanted coordinate transformation. Thus χ_r has to be chosen such that $\Pi^r_{nr}(H\circ\mathcal{T}_r) = 0$. In order to write explicitly such an equation, remark that, since the Lie transform generated by a smoothing function is almost smooth, after any number of coordinate transformations the Hamiltonian has the form (4.27), (4.29), (4.30).

In order to compute the coefficients to be put equal to zero we work in $\mathcal{V}^{\infty}$, in which also an almost smooth map can be expanded in Taylor series. Given two functions $\chi(M,\phi)$ and $F(M,\phi)$ we denote by $\{\chi;F\}^{st}$ the Poisson bracket of the two functions *computed at constant M, i.e. as if M where independent of ϕ*. Similarly we will denote by $X^{st}_{\chi}(M,\phi)$ the Hamiltonian vector field of χ computed as if M where independent of ϕ.

We first study the simpler case in which $r \geq 3$.

Lemma 5.3. *Assume that $\chi \in C^{\infty}(\mathbb{R}^n;\mathrm{Gen}_k)$ is a homogeneous polynomial of degree $r \geq 3$, then one has*

$$\Pi^r_{nr}(H\circ\mathcal{T}_r) = \Pi_{nr}\left[\{H_{L0}+H_{L1};\chi\}^{st} + \frac{\partial D}{\partial M_j}(M)\langle A_j\phi; X^{st}_{\chi}\rangle\right] + \Pi^r_{nr}H_N \tag{5.7}$$

Furthermore one has $\Pi^{r_1}_{nr}(H\circ\mathcal{T}_r) = \Pi^{r_1}_{nr}(H)$, $\forall r_1 < r$.

Proof. First remark that, by lemma 3.19 and remark 4.7, one has

$$\mathcal{T}_r(\phi) = \phi + X^{st}_{\chi}(\phi) + \mathcal{O}(|\phi|^r) \ ,$$

and therefore $H_N\circ\mathcal{T}_r = H_N + \mathcal{O}(|\phi|^{r+1})$, which shows that $\Pi^{r_1}_{nr}(H_N\circ\mathcal{T}_r) = \Pi^{r_1}_{nr}(H_N)$ $\forall r_1 \leq r$. We come to $H_L\circ\mathcal{T}_r$. Denote $\phi' = \mathcal{T}_r(\phi)$ and

$$M'_j = M_j\circ\mathcal{T}_r = M_j + \langle A_j\phi; X^{st}_{\chi}(M,\phi)\rangle + \mathcal{O}(|\phi|^{r+1}) \ ,$$

then one has

$$
\begin{aligned}
H_L(M',\phi') &= H_{L0}(\phi) + \mathrm{d}H_{L0}(\phi)X^{st}_\chi(M,\phi) + \mathcal{O}(|\phi|^{r+1}) \\
&\quad + H_{L1}(M,\phi') + \mathcal{O}(|\phi|^{r+1}) + D(M) \\
&\quad + \frac{\partial D}{\partial M_j}(M)\langle A_j\phi; X^{st}_\chi(M,\phi)\rangle + \mathcal{O}(|\phi|^{r+1}) \\
&= H_{L0}(\phi) + \mathrm{d}H_{L0}(\phi)X^{st}_\chi(M,\phi) + \mathrm{d}H_{L1}(M,\phi)X^{st}_\chi(\phi) \\
&\quad + D(M) + \frac{\partial D}{\partial M_j}(M)\langle A_j\phi; X^{st}_\chi(M,\phi)\rangle + \mathcal{O}(|\phi|^{r+1})
\end{aligned}
\tag{5.8}
$$

where the differentials are computed at constant M. The application of Π^r_{nr} and the remark that $\mathrm{d}H_{L1}X^{st}_\chi = \{H_{L1};\chi\}^{st}$ give the result. □

Lemma 5.4. *The function $\Pi^r_{nr}H_N$ is smoothing, and thus admits the representation* (5.5) *with $\Phi_{\mu\nu} \in \mathcal{W}^\infty$.*

Proof. Consider $H_N(\phi_d + \phi_c)$, and remark that only $H_N(\phi_d)$ and $\mathrm{d}H_N(\phi_d)\phi_c$ contribute to $\Pi_{nr}H_N$. Now, $H_N(\phi_d)$ is clearly smooth. To compute the other term we use formula (4.30). Clearly the only term to be discussed is the one coming from H^3_P. In order to compute it we use the definition (4.6) of H^3_P. Compute first the differential with respect to ϕ:

$$
\mathrm{d}H^3_P(\eta,\phi_d)\Phi = \mathrm{d}H_P(\eta+\phi_d)\Phi - \left[\mathrm{d}H_P(\eta)\Phi + \mathrm{d}^2H_P(\eta)(\phi_d,\Phi)\right] \ ; \tag{5.9}
$$

substituting $\Phi = \phi_c + \mathrm{d}S(\phi_d)\phi_c$ one gets the formula for the term we have to compute. Here we denoted $S := S^1_1 + S^0_2$. Then, by assumption (P2) the expression (5.9) is smoothing. □

Lemma 5.5. *For any $r \geq 3$ there exists χ_r such that, denoting by $\mathcal{T}_r$ the corresponding Lie transform, one has $\Pi^r_{nr}(H \circ \mathcal{T}_r) = 0$.*

Proof. We study the map

$$
C^s(\mathbb{R}^n;\mathrm{Gen}_k) \ni \chi \mapsto \mathcal{L}\chi := \Pi^r_{nr}\left(H_L \circ \mathcal{T}_r\right) \tag{5.10}
$$

$$
= \Pi_{nr}\left[\{H_{L0}+H_{L1};\chi\}^{st} + \frac{\partial D}{\partial M_j}(M)\langle A_j\phi; X^{st}_\chi\rangle\right] \in C^s(\mathbb{R}^n;\mathrm{Gen}_{k-d_0}) \ . \tag{5.11}
$$

$\mathcal{L}$ is a relatively bounded perturbation of the linear operator

$$
\mathrm{Gen}_k \ni \chi \mapsto \mathcal{L}_0\chi := \Pi_{nr}\left\{H_{L0};\chi\right\}^{st} \in \mathrm{Gen}_{k-d_0} \ , \tag{5.12}
$$

furthermore $\mathcal{L} - \mathcal{L}_0$ has a norm of order M. So we study $\mathcal{L}_0^{-1}$. To this end remark that one has

$$
\mathcal{L}_0\chi = \sum_{\mu\nu} -\mathrm{i}\omega\cdot(\mu-\nu)\chi_{\mu\nu}\xi^\mu\bar{\xi}^\nu \tag{5.13}
$$

$$
+\sum_{\mu\nu}\left[\mathrm{i}\omega\cdot(\nu-\mu)\langle E\Phi_{\mu\nu};\phi_c\rangle - \langle EL_c\Phi_{\mu\nu};\phi_c\rangle\right]\xi^\mu\bar{\xi}^\nu \ , \tag{5.14}
$$

so that one has that $\chi = \mathcal{L}_0^{-1}\Pi_{nr}F$ is given by

$$\chi_{\mu\nu} = \frac{F_{\mu\nu}}{-\mathrm{i}\omega\cdot(\mu-\nu)} \ , \quad \text{for } \omega\cdot(\mu-\nu)\neq 0 \tag{5.15}$$

$$\Phi_{\mu\nu} = R_{L_c}(\mathrm{i}\omega\cdot(\nu-\mu))\Phi^F_{\mu\nu} \ . \tag{5.16}$$

By (L1), the resolvent maps $\mathcal{W}^{k-d_0}$ into $\mathcal{W}^k$, thus it is regularizing, therefore equations (5.15), (5.16) show that the inverse of $\mathcal{L}_0$ is smooth as a map from Gen_{k-d_0} to Gen_k. So $\mathcal{L}$ can be inverted by Neumann formula, giving the result. □

We come now to the more complicated case $r = 2$. One has $X^{st}_\chi(\phi) = T(M)\phi$ with $T(M)$ a linear smoothing operator smoothly dependent on M. Furthermore, remark that, by the proof of lemma 3.19, one has

$$\phi' = \mathcal{T}_2(\phi) = \mathrm{e}^{T(M)}\phi + \mathcal{O}(|\phi|^3) \Longrightarrow \Pi^2_{nr}(H_L\circ\mathcal{T}_2) = \Pi^2_{nr}(H_L\circ \mathrm{e}^{T(M)}) \ . \tag{5.17}$$

Thus

$$\Pi^2_{nr}\left(H_L\circ\mathcal{T}_2\right) = \Pi^2_{nr}\Big(\left\{\chi_2;H_{L0}\right\}^{st} + H_{L1}(M,\phi) \tag{5.18}$$

$$+H_{L1}(M(\mathrm{e}^{T(M)}\phi),\mathrm{e}^{T(M)}\phi) - H_{L1}(M,\phi) + D(M(\mathrm{e}^{T(M)}\phi)) - D(M) \tag{5.19}$$

$$+H_{L0}(\mathrm{e}^{T(M)}\phi) - \Big[H_{L0}(\phi) + \left\{\chi_0;H_{L0}\right\}^{st}\Big]\Big) \tag{5.20}$$

which is a small perturbation of the first line. We will solve $\Pi^2_{nr}\left(H_L\circ\mathcal{T}_2\right) = 0$ using the implicit function theorem, working perturbatively with respect to the first line. First we need to estimate the other lines. To this end we need the following lemma.

Lemma 5.6. *On the space of the functions $\chi\in\mathrm{Gen}_k$ homogeneous of degree 2, the norm (5.4) is equivalent to the norm of X^{st}_χ as a linear operator from $\mathcal{H}^{-k}$ to $\mathcal{H}^k$.*

Proof. One has

$$X^{st}_\chi\begin{pmatrix}\xi_k\\ \phi_c\end{pmatrix} = \begin{pmatrix}-\mathrm{i}\sum_{|\mu|+|\nu|=2}\chi_{\mu\nu}\nu_k\frac{\xi^\mu\bar\xi^\nu}{\xi_k} + \sum_{|\mu|+|\nu|=1}\nu_k\frac{\xi^\mu\bar\xi^\nu}{\xi_k}\left\langle E\Phi_{\mu\nu},\phi_c\right\rangle \\ \sum_{|\mu|+|\nu|=1}\xi^\mu\bar\xi^\nu\Phi_{\mu\nu} \ .\end{pmatrix} \tag{5.21}$$

so it is clear that the norm of such a linear operator is controlled by the norm (5.4). We have also to prove that the norm of a single function $\Phi_{\mu\nu}$ (and the modulus of the coefficients $\chi_{\mu\nu}$) is controlled by the norm of the linear operator. To see this, remark that for example taking $\phi_c = 0$ and $\xi_1 = 1$, $\xi_k = 0$ for $k\neq 1$, one gets $T(\xi;\phi_c) = (\chi_{\mu^1\nu^1},\Phi_{\mu^2\nu^2})$ with $\mu^1 = (1,0,0,0...)$ and so on. Thus the norm of each of the two components is controlled by the operator norm of X^{st}_χ. □

Lemma 5.7. *There exists χ_2 of the form (5.3) with $r=2$, such that $\Pi^2_{nr}(H_L\circ\mathcal{T}_2) = 0$*

Proof. Define

$$G_1(M,\chi) := \Pi_{nr}\left((5.19) + (5.20)\right) \tag{5.22}$$

equation $\Pi^2_{nr}(H_L \circ \mathcal{T}_2) = 0$ coincides with

$$0 = \mathcal{L}_0\chi + \Pi_{nr}H_{L1} + G_1(M,\chi)\ , \quad \mathcal{L}_0\chi := \{H_{L0};\chi\}\ . \tag{5.23}$$

Since by the same reasoning of the proof of lemma 5.5, $\mathcal{L}_0^{-1}$ is bounded as an operator from Gen_k to Gen_{k+d_0}, and the norm of $G_1(M,\chi)$ is smaller then $C|M|\,\|\chi\|$, one can apply the implicit function theorem to (5.23), getting the result. □

This concludes the proof of theorem 5.2.

6 Dispersive Estimates

From now on we restrict our setting to the situation of NLS, but we try to write clearly the assumptions we use, in order to make easy the application to other models. Thus, from now on the scale $\mathcal{H}^k$ will be that of the weighted Sobolev space $H^{k,l}$ (where the measure is $\langle x\rangle^l\,dx$) and $\mathcal{H}^\infty$ =Schwartz space. When we write only one index we mean the standard Sobolev spaces without weight. We will also use the Lebesgue spaces L^p and assume $d_0 = 2$.

In this section we will systematically use the notation $a \preceq b$ to mean "there exists a positive C, independent of all the relevant quantities, s.t. $a \le Cb$".

Given functions $w^j(.) \in C^0([0,T],\mathbb{R}^n)$, consider the time dependent linear operator

$$L(t) := P_cJ\left[A_0 + V_0 - w^j(t)A_j - \lambda^j(p_0)A_j\right]P_c = L_c - w^j(t)J\mathcal{A}_j\ . \tag{6.1}$$

and denote by $\mathcal{U}(t,s)$ the evolution operator of the equation $\dot\phi = L(t)\phi$; we assume that there exists $\epsilon > 0$ such that, if $|w^j(t)| < \epsilon$ then the following Strichartz estimates hold

(St.1)

$$\|\mathcal{U}(t,0)\phi_c\|_{L^2_tL^6_x} \preceq \|\phi_c\|_{L^2_x}\ , \tag{6.2}$$

$$\left\|\int_0^t \mathcal{U}(t,s)F(s)ds\right\|_{L^2_tL^6_x} \preceq \|F\|_{L^2_tL^{6/5}_x}\ . \tag{6.3}$$

For $\rho \in \pm\mathrm{i}(\Omega,\infty)$ denote $R^\pm_{L_c}(\rho) := \lim_{\epsilon\to0^+}(L_c - \rho \pm \epsilon)^{-1}$; then we assume

(St.2) For any $\Phi \in \mathcal{W}^\infty$, any complex valued function $h(.) \in L^2_t$, any $\rho \in \pm\mathrm{i}(\Omega,\infty)$ one has

$$\left\|\langle x\rangle^{-a}\mathcal{U}(t,0)R^\pm_{L_c}(\rho)\Phi\right\|_{L^2_x} \preceq \frac{\|\langle x\rangle^a\Phi\|_{L^2_x}}{\langle t\rangle^{3/2}}\ , \tag{6.4}$$

$$\left\|\int_0^t \mathcal{U}(t,s)h(s)R^\pm_{L_c}(\rho)\Phi ds\right\|_{L^2_tL^{2,-a}_x} \preceq \|h\|_{L^2_t}\|\Phi\|_{L^{2,a}_x}\ . \tag{6.5}$$

where a is a sufficiently large constant.

(St.3) There exists a s.t., for any $k>0$ and any $\Phi \in \mathcal{W}^\infty$, $\rho \in \pm(\Omega,\infty)$ one has

$$R^{\pm}_{L_c}(\mathrm{i}\rho)\Phi \in H^{k,-a} \tag{6.6}$$

and, for any $k,a>0$ one has

$$[JV_0, JA_j] : H^{k,-a} \to H^{k,a} \ . \tag{6.7}$$

Finally we need some smoothness of the vector field of H_P

(P3) The map X^2_P defined in (4.8) is continuous as a map

$$H^{k,l} \times (H^1 \cap L^6) \ni (\eta,\phi) \mapsto X^2_P \in L^{6/5} \ ,$$

with k,l sufficiently large; furthermore, for $\phi \in H^1$, with $\|\phi\|_{H^1} \le \epsilon$, one has

$$\left\|X^2_P(\eta;\phi)\right\|_{L^{6/5}} \preceq \epsilon C \left\|\phi\right\|_{L^6} \ , \tag{6.8}$$

with $C = C(\epsilon, \|\eta\|_{H^{k,l}})$.

The map

$$\mathcal{H}^\infty \times L^6 \ni (\eta,\phi) \mapsto \mathrm{d}X_P(\eta)\phi \in L^{6/5} \tag{6.9}$$

is C^1.

The main result of this section is the following theorem

Theorem 6.1. *Consider the Hamiltonian* (4.27) *and assume it is in normal form at order* $2r_t$*. Assume also that the Fermi Golden Rule* (6.52) *below holds. Let* $\phi(t)$ *be a solution of the corresponding Hamilton equations with an initial datum* ϕ_0 *fulfilling*

$$\|\phi_0\|_{H^1} \le \epsilon \tag{6.10}$$

and ϵ *small enough, then one has*

$$\|\phi_c(t)\|_{L^2_t L^6_x} \preceq \epsilon \tag{6.11}$$

$$\omega\cdot\mu > \Omega \ , \Longrightarrow \ \|\xi^\mu(.)\|_{L^2_t} \preceq \epsilon \ . \tag{6.12}$$

The rest of the section will be devoted to the proof of this theorem.

6.1 Estimate of the continuous variable

Given a Hamiltonian of the form (4.27), in normal form at order $2r_t$ we study the solution of the corresponding Hamilton equations. It will be denoted by $\phi(t)$.

Remark 6.2. Let G be a map of the form $G(\phi) = \phi + S(\phi)$, with a smoothing S. Consider the Hamiltonian $H^3_P(\eta, G(\phi))$, with a fixed η. Then one has

$$J\nabla(H^3_P \circ G) = J\left[dG(\phi)\right]^* E X^2_P(\eta, G(\phi)) \ , \tag{6.13}$$

where $\mathrm{d}G(\phi)^*$ is the adjoint of the operator $\mathrm{d}G(\phi)$.

Remark 6.3. Using orbital stability (which follows from (L2)) one has that, given an initial datum with $\|\phi\|_{H^1} \leq C\epsilon$, then

$$|M_j(t)| \leq C\epsilon^2 \ , \quad \forall t \ .$$

Remark 6.4. One has

$$\begin{aligned} &X_P^2(\eta; G(\phi_d+\phi_c)) - X_P^2(\eta; G(\phi_d)) \\ &\qquad = X_P^2(\eta+\phi_d+S(\phi_d); \phi_c + S(\phi_d+\phi_c) - S(\phi_d)) \\ &\qquad + \left[\mathrm{d}X_P(\eta+\phi_d+S(\phi_d)) - \mathrm{d}X_P(\eta)\right](\phi_c + S(\phi_d+\phi_c) - S(\phi_d)) \ . \end{aligned} \tag{6.14}$$

Lemma 6.5. *Let G be as in remark 6.2, fix $\eta \in \mathcal{H}^\infty$ and consider a Hamiltonian function of the form $H_P^3(\eta; G(\phi)) + R_3^0$. Assume it is in normal form at order $2r_t$. Let X be its Hamiltonian vector field. Assume that for some $T > 0$ the functions $\phi_c(t)$ and $\xi(t)$ fulfill the estimates*

$$\|\phi_c\|_{L^2_t[0,T]L^6_x} \leq C_1\epsilon \ , \tag{6.15}$$

$$\|\xi^\mu\|_{L^2_t[0,T]} \leq C_2\epsilon \ , \quad \forall \mu \in K := \{\mu \ : \ \omega\cdot\mu > \Omega \ , \ |\mu| \leq r_t\} \ , \tag{6.16}$$

$$\|\phi_c(t)\|_{H^1_x} \preceq \epsilon \ , \quad |\xi(t)| \preceq \epsilon \ ; \tag{6.17}$$

then there exists C independent of ϵ, C_1, C_2 such that one has

$$\|P_cX(\phi)\|_{L^2_t[0,T]L^{6/5}_x} \leq \epsilon C(C_2 + \epsilon C_1) \ . \tag{6.18}$$

Proof. Write $X(\phi_c+\phi_d) = X(\phi_d) + X(\phi_c+\phi_d) - X(\phi_d)$. From (P2) it is clear that $X(\phi_d(\xi,\bar{\xi}))$ is a C^∞ function taking values in $\mathcal{V}^\infty$. Furthermore, write $P_cX(\phi_d) = P_cX_{\leq r_t}(\phi_d) + P_cX_{>r_t}(\phi_d)$, where $P_cX_{\leq r_t}(\phi_d)$ is the Taylor expansion truncated at order $2r_t$, which therefore (due to the fact that the system is in normal form) contains only monomials of the form $\Phi_{\mu\nu}\xi^\mu\bar{\xi}^\nu$ with $\Phi_{\mu\nu} \in \mathcal{W}^\infty$ and $|\omega\cdot(\mu-\nu)| > \Omega$. This implies in particular

$$\|P_cX_{\leq r_t}(\phi_d)\|_{L^{6/5}_x} \preceq \sum_{|\mu|\leq 2r_t \ , |\omega\cdot\mu|>\Omega} |\xi^\mu| \ . \tag{6.19}$$

It follows from (6.16) that

$$\|P_cX_{\leq r_t}(\phi_d)\|_{L^2_t[0,T]L^{6/5}_x} \leq CC_2\epsilon \ .$$

Concerning $X_{>r_t}$, just remark that, by the formula for the remainder of the Taylor expansion, one has

$$\|P_cX_{>r_t}(\phi_d)\|_{L^{6/5}_x} \preceq \left(|\xi_1|^2 + ... + |\xi_K|^2\right)^{(r_t+1)/2} \ .$$

Controlling the r.h.s. through $\|\xi^\mu\|_{L^2_t}$, $\mu \in K$, one gets the thesis

We have now to estimate $P_cX(\phi_c+\phi_d) - P_cX(\phi_d)$. By remark 6.2, it is the sum of a smoothing term coming from R_3^0 and of the quantity

$$[\mathrm{d}G^*(\phi_d+\phi_c) - \mathrm{d}G^*(\phi_d)]EX_P^2(\eta_{p_0-M}; G(\phi_d+\phi_c)) \tag{6.20}$$

$$+\mathrm{d}G^*(\phi_d)E\left[X_P^2(\eta; G(\phi_d+\phi_c)) - X_P^2(\eta; G(\phi_d))\right] \ . \tag{6.21}$$

Since $\mathrm{d}G^*(\phi_d+\phi_c)-\mathrm{d}G^*(\phi_d)=\mathrm{d}S^*(\phi_d+\phi_c)-\mathrm{d}S^*(\phi_d)$, (where we used the notations of lemma 6.2) one has

$$\|\mathrm{d}G^*(\phi_d+\phi_c)-\mathrm{d}G^*(\phi_d)\|_{B(H^{-k_1,-l_1};H^{k_2,l_2})}\preceq\|\phi_c\|_{H^{-k_1,-l_1}}\preceq\|\phi_c\|_{L^6}\ ,$$

and therefore,

$$\begin{aligned}\left\|[\mathrm{d}G^*(\phi_d+\phi_c)-\mathrm{d}G^*(\phi_d)]EX_P^2(\eta_{p_0-M};G(\phi_d+\phi_c))\right\|_{L^{6/5}}\\ \preceq\|\phi_c\|_{L^6}\left\|X_P^2(\eta;G(\phi_c+\phi_d))\right\|_{L^{6/5}}\\ \preceq\|\phi_c\|_{L^6}\,\epsilon\,\|G(\phi_c+\phi_d)\|_{L^6}\preceq\epsilon\,\|\phi_c\|_{H^1}\|\phi_c\|_{L^6}\preceq\epsilon^2\|\phi_c\|_{L^6}\ .\end{aligned}\tag{6.22}$$

In order to estimate (6.21), we exploit Remark 6.4 which gives

$$\begin{aligned}\left\|X_P^2(\eta;G(\phi_d+\phi_c))-X_P^2(\eta;G(\phi_d))\right\|_{L^{6/5}}\\ \preceq\epsilon\,\|\phi_c+S(\phi_c+\phi_d)-S(\phi_d)\|_{L^6}+\|G(\phi_d)\|_{H^{k,l}}\|\phi_c\|_{L^6}\preceq\epsilon\,\|\phi_c\|_{L^6}\ .\end{aligned}\tag{6.23}$$

Adding the trivial estimate of $\mathrm{d}G^*(\phi_d)$ one gets the thesis. □

We are now ready to give the estimate of the continuous variable ϕ_c.

Lemma 6.6. *Let $\phi(t)$ be a solution of the considered system. Assume that the initial datum ϕ fulfills $\|\phi\|_{H^1}\leq\epsilon$ for some ϵ small enough. Assume that, for some $T>0$, the a priori estimates* (6.15), (6.16) *and* (6.17) *are fulfilled then ϕ_c fulfills the following estimate*

$$\|\phi_c(t)\|_{L^2_t[0,T]L^6_x}\leq C\epsilon(C_2+\epsilon C_1)\ .\tag{6.24}$$

Proof. First, the equation for ϕ_c has the form

$$\dot{\phi}_c=L(t)\phi_c+P_cJ[(V_M-V_0)\phi+(S_1^1)_{lin}\phi]+P_cX_N(\phi)\ .\tag{6.25}$$

We also denoted by $(S_1^1)_{lin}$ a linear smoothing operator whose norm tends to zero when $M\to 0$. Remark that, since the system is in normal form at order r_t, the second term is independent of ϕ_d. Thus (6.25) is equivalent to

$$\dot{\phi}_c=L(t)\phi_c+P_cJ[(V_M-V_0)\phi_c+(S_1^1)_{lin}\phi_c]+P_cX_N(\phi)\ .\tag{6.26}$$

We use Duhamel principle to write its solution in the form $\phi_c(t)=I_1+I_2+I_3+I_4$, where

$$I_1:=\mathcal{U}(t,0)\phi_c(0)\ ,\quad I_2:=\int_0^t\mathcal{U}(t,s)P_cJ(V_M-V_0)\phi_c(s)\mathrm{d}s\ ,\tag{6.27}$$

$$I_3:=\int_0^t\mathcal{U}(t,s)P_c(S_1^1)_{lin}\phi_c(s)\mathrm{d}s\ ,\quad I_4:=\int_0^t\mathcal{U}(t,s)P_cX(\phi(s))\mathrm{d}s\ .\tag{6.28}$$

The estimates of I_1 and of I_4 are an immediate consequence of (St.1) and lemma 6.5, which give

$$\|I_1\|_{L^2_t[0,T]L^6_x}\preceq\epsilon\ ,\quad\|I_4\|_{L^2_t[0,T]L^6_x}\preceq\epsilon(C_2+\epsilon C_1)\ .$$

Concerning I_2 we have, by (P3)

$$\|(V_M - V_0)\phi_c(s)\|_{L_x^{6/5}} \preceq |M| \, \|\phi_c\|_{L_x^6} \preceq \epsilon^2 \, \|\phi_c\|_{L_x^6} \ ,$$

from which $\|I_2\|_{L_t^2[0,T]L_x^6} \preceq \epsilon^3 C_1$. Similarly, I_3 is estimated using

$$\left\|P_c(S_1^1(M))_{lin}\phi_c\right\|_{L_x^{6/5}} \preceq \left\|(S_1^1(M))_{lin}\phi_c\right\|_{H^{k,l}} \preceq |M| \, \|\phi_c\|_{H^{-k,-l}} \preceq \epsilon^2 \, \|\phi_c\|_{L^6} \ .$$

from which $\|I_3\|_{L_t^2[0,T]L_x^6} \preceq \epsilon^3 C_1$, and the thesis. $\square$

6.2 A further step of normalization

Consider again the Hamiltonian in normal form at order $2r_t$, we rewrite it in a form suitable for the forthcoming developments. First write

$$H_{re}(\phi_d, \phi_c) := H_N(\phi_d + \phi_c) - H_N(\phi_d) - \mathrm{d}H_N(\phi_d)\phi_c \ , \tag{6.29}$$

and

$$H_{Nd} := H_N(\phi_d(\xi, \bar{\xi})) - Z_0(\xi, \bar{\xi}) \ , \quad H_{Nc} := \mathrm{d}H_N(\phi_d)\phi_c - Z_1(\xi, \bar{\xi}, \phi_c) \ , \tag{6.30}$$

where Z_0 is the Taylor expansion of $H_N(\phi_d(\xi, \bar{\xi}))$ truncated at order $2r_t$, and we defined

$$Z_1(\xi, \bar{\xi}, \phi_c) := \langle EG, \phi_c \rangle + \langle E\bar{G}, \phi_c \rangle \ , \tag{6.31}$$

$$G := \sum_{\nu \in \mathcal{K}} \bar{\xi}^\nu \Phi_\nu \ , \quad \bar{G} = \overline{\sum_{\nu \in \mathcal{K}} \bar{\xi}^\nu \Phi_\nu}, \tag{6.32}$$

$\Phi_\nu \in \mathcal{W}^\infty$ and

$$\mathcal{K} := \{\nu \ : \ 2 \le |\nu| \le 2r_t \ , \quad \omega \cdot \nu > \Omega\} \ . \tag{6.33}$$

Denote also $\mathcal{R} := H_{Nd} + H_{Nc} + H_{re}$, then the Hamilton equations of the system can be written in the form

$$\dot{\phi}_c = L_c\phi_c + J\nabla_{\phi_c} H_{L1} + G + \bar{G} + w^j(M, \xi, \phi_c) J\mathcal{A}_j\phi_c + J\nabla_{\phi_c}\mathcal{R}(M, \xi, \phi_c) \ , \tag{6.34}$$

$$\dot{\xi}_k = -\mathrm{i}\omega_k\xi_k - \mathrm{i}\frac{\partial H_{L1}}{\partial \bar{\xi}_k} - \mathrm{i}\frac{\partial Z_0}{\partial \bar{\xi}_k} - \mathrm{i}\left\langle E\frac{\partial G}{\partial \bar{\xi}_k}, \phi_c \right\rangle - \mathrm{i}\frac{\partial \mathcal{R}}{\partial \bar{\xi}_k} \ , \tag{6.35}$$

$$w^j(M, \xi, \phi_c) := \frac{\partial H_N}{\partial M_j} + \frac{\partial H_{L1}}{\partial M_j} + \frac{\partial D}{\partial M_j} \ , \tag{6.36}$$

and the gradient ∇_{ϕ_c} is computed at constant M. We look now for functions Y_ν such that the new variable g defined by

$$g := \phi_c + Y + \bar{Y} \ , \quad Y = \sum_\nu Y_\nu \bar{\xi}^\nu \tag{6.37}$$

is decoupled up to higher order terms from the discrete variables. Substitution into equation (6.34) yields

$$\dot{g} = L_c g + \sum_\nu \left(\Phi_\nu + \mathrm{i}\nu\cdot\omega Y_\nu - L_c Y_\nu\right)\bar{\xi}^\nu \tag{6.38}$$

$$+ \sum_\nu \left(\bar{\Phi}_\nu - \mathrm{i}\nu\cdot\omega \bar{Y}_\nu - L_c \bar{Y}_\nu\right)\xi^\nu + \text{h.o.t.} \tag{6.39}$$

where the h.o.t. will be explicitly computed below. In order to kill the main terms define

$$Y_\nu = R^+_{L_c}(\mathrm{i}\omega\cdot\nu)\Phi_\nu \quad \text{and} \quad \bar{Y}_\nu = \overline{R^+_{L_c}(\mathrm{i}\omega\cdot\nu)\Phi_\nu} = R^-_{L_c}(-\mathrm{i}\omega\cdot\nu)\bar{\Phi}_\nu\ . \tag{6.40}$$

We substitute (6.37) into (6.35). Then, using (6.40), we get

$$\dot{\xi}_k = -\mathrm{i}\omega_k\xi_k - \mathrm{i}\frac{\partial Z_0}{\partial\bar{\xi}_k} - \mathrm{i}\frac{\partial H_{L1}}{\partial\bar{\xi}_k} + \mathcal{G}_{0,k}(\xi) - \mathrm{i}\left\langle E\frac{\partial G}{\partial\bar{\xi}_k}; g\right\rangle - \mathrm{i}\frac{\partial\mathcal{R}}{\partial\bar{\xi}_k}\ , \tag{6.41}$$

$$\mathcal{G}_{0,k}(\xi) := \mathrm{i}\sum_{\nu\in\mathcal{K},\nu'\in\mathcal{K}}\left(\frac{\xi^{\nu'}\bar{\xi}^\nu}{\bar{\xi}_k}\nu_k c_{\nu\nu'} + \frac{\bar{\xi}^{\nu'}\bar{\xi}^\nu}{\bar{\xi}_k}\nu_k b_{\nu\nu'}\right)\ , \tag{6.42}$$

$$c_{\nu\nu'} := \left\langle E\Phi_\nu, \bar{Y}_{\nu'}\right\rangle, b_{\nu\nu'} := \left\langle E\Phi_\nu, Y_{\nu'}\right\rangle\ . \tag{6.43}$$

Following the standard ideas of normal form theory, we look for a change of variables of the form $\eta_j = \xi_j + \Delta_j(\xi)$ which moves to higher order the nonresonant terms. The choice

$$\Delta_j(\xi) := \sum_{\substack{\mu\in\mathcal{K},\,\nu\in\mathcal{K}\\ \omega\cdot(\mu-\nu)\neq 0}}\left[\frac{1}{\mathrm{i}\omega\cdot(\mu-\nu)}\frac{\xi^\mu\bar{\xi}^\nu}{\bar{\xi}_j}\nu_j c_{\nu\mu} + \frac{1}{-\mathrm{i}\omega\cdot(\mu+\nu)}\frac{\bar{\xi}^\mu\bar{\xi}^\nu}{\bar{\xi}_j}\nu_j b_{\nu\mu}\right] \tag{6.44}$$

transforms (6.41) into the system $\dot{\eta}_k = \Xi_k(\eta,\bar{\eta}) + \mathcal{E}_k(t)$ where

$$\Xi_k(\eta,\bar{\eta}) := -\mathrm{i}\omega_k\eta_k - \mathrm{i}\frac{\partial Z_0}{\partial\bar{\eta}_k} - \mathrm{i}\frac{\partial H_{L1}}{\partial\bar{\xi}_k} + \mathcal{N}_k(\eta) \tag{6.45}$$

$$\mathcal{N}_k(\eta) := \mathrm{i}\sum_{\substack{\mu\in\mathcal{K},\,\nu\in\mathcal{K}\\ \omega\cdot(\mu-\nu)=0}}\frac{\eta^\mu\bar{\eta}^\nu}{\bar{\eta}_k}\nu_k c_{\nu\mu}\ , \tag{6.46}$$

and $\mathcal{E}_j(t)$ is a remainder term whose expression is explicitly given by

$$\begin{aligned}\mathcal{E}_j :=\ & X^{L1}_j(\xi) - X^{L1}_j(\eta) + \mathcal{G}_{0j}(\xi) - \mathcal{G}_{0j}(\eta) + X^N_j \\ & + \sum_k\left(\frac{\partial\Delta_j}{\partial\xi_k}(\xi)\left[X^{L1}_k + \mathcal{G}_{0k}(\xi) + X^N_k(\xi)\right] + \frac{\partial\Delta_j}{\partial\bar{\xi}_k}(\xi)\left[\bar{X}^{L1}_k + \bar{\mathcal{G}}_{0k}(\xi) + \bar{X}^N_k(\xi)\right]\right) \\ & - \sum_k\left(\frac{\partial\Delta_j}{\partial\xi_k}(\xi)\mathrm{i}\omega_k\xi_k + \frac{\partial\Delta_j}{\partial\xi_k}(\eta)\mathrm{i}\omega_k\eta_k + \frac{\partial\Delta_j}{\partial\bar{\xi}_k}(\xi)\mathrm{i}\omega_k\bar{\xi}_k - \frac{\partial\Delta_j}{\partial\bar{\xi}_k}(\eta)\mathrm{i}\omega_k\bar{\eta}_k\right)\ ,\end{aligned} \tag{6.47}$$

and we denoted

$$X_k^{L1} := -\mathrm{i}\frac{\partial Z_0}{\partial \bar{\xi}_k} - \mathrm{i}\frac{\partial H_{L1}}{\partial \bar{\xi}_k} \ , \quad X_k^N := \mathrm{i}\frac{\partial \mathcal{R}}{\partial \bar{\xi}_k} - \mathrm{i}\left\langle E\frac{\partial G}{\partial \bar{\xi}_k}; g\right\rangle \ .$$

The key point is that the considered system for η is no more conservative. To see this we compute the Lie derivative of $H_{0L\xi} \equiv \sum_k \omega_k \left|\eta_k\right|^2$ with respect to Ξ.

We partition $\mathcal{K}$ into "resonant sets". Define

$$\Lambda := \{\lambda \in \mathbb{R} \ : \ \lambda = \omega \cdot \mu \ , \quad \mu \in \mathcal{K}\}$$

and, for $\lambda \in \Lambda$, define

$$\mathcal{K}_\lambda := \{\mu \in \mathcal{K} \ : \ \omega \cdot \mu = \lambda\} \tag{6.48}$$

$$F_\lambda(\eta) := \sum_{\mu \in \mathcal{K}_\lambda} \Phi_\mu \bar{\eta}^\mu \in \mathcal{W}^\infty \ . \tag{6.49}$$

Remark 6.7. One has

$$\pounds_\Xi H_{0L\xi} = -\operatorname{Im}\left(\sum_{\lambda \in \Lambda} \lambda \left\langle E\overline{F_\lambda(\eta)}; R^+_{L_c}(\mathrm{i}\lambda) F_\lambda(\eta)\right\rangle\right) \ . \tag{6.50}$$

Furthermore, using formally the formula $(x - \mathrm{i}0)^{-1} = PV(1/x) + \mathrm{i}\pi\delta(x)$ in order to compute (formally) $R^+_{L_c}(\mathrm{i}\lambda)$, one realizes that if there are no convegence problems, one has

$$\langle E\bar{\Phi}; R^+_{L_c}(\mathrm{i}\lambda)\Phi\rangle \geq 0 \ , \quad \forall \Phi \in \mathcal{W}^\infty \ . \tag{6.51}$$

In typical cases (e.g. in NLS), (6.51) is obtained by using the wave operator in order to conjugate L_c and JA_0, and exploiting the result by [Yaj95, Cuc01] according to which the wave operator leaves invariant the L^p spaces.

We are ready to state the Fermi Golden Rule, which essentially states that the quantity (6.50) is nondegenerate; to this end denote

$$b_\lambda(\eta) := \operatorname{Im}\left(\sum_{\lambda \in \Lambda} \lambda \left\langle E\overline{F_\lambda(\eta)}; R^+_{L_c}(\mathrm{i}\lambda) F_\lambda(\eta)\right\rangle\right)$$

(FGR) there exists a positive constant C and a sufficiently small $\delta_0 > 0$ such that for all $|\eta| < \delta_0$

$$\sum_{\lambda \in \Lambda} \lambda b_\lambda(\eta) \geq C \sum_{\mu \in \mathcal{K}} |\eta^\mu|^2 \ . \tag{6.52}$$

Remark 6.8. This is version of the FGR is essentially identical to that used in [GW08]. It is easy to see that in the nonresonant case $\#\mathcal{K}_\lambda = 1 \ \forall \lambda \in \Lambda$, (6.52) is equivalent to the assumption that a finite number of coefficients is different from zero (see [BC11] condition (H7")).

6.3 Estimate of the variables g, ξ, η.

In order to estimate the variable g we need the following lemma

Lemma 6.9. *For any $\Phi \in \mathcal{W}^\infty$ and any $\rho \in \sigma_c(L_c)$, there exists a $\Psi \in \mathcal{W}^\infty$, linearly dependent on Φ, such that one has*

$$\left[R^{\pm}_{L_c}(\rho), JA_j\right]\Phi = R^{\pm}_{L_c}(\rho)\Psi \ . \tag{6.53}$$

Proof. To start with take $\rho \notin \sigma(L_c)$. A simple computation shows that (omitting ρ) the l.h.s. of (6.53) is given by $R_{L_c}[JV_0; JA_j]R_{L_c}\Phi$, from which, using (St.3) and taking the limit $\rho \to \sigma_c$, the thesis follows. □

Lemma 6.10. *Under the same assumptions of lemma 6.6, g fulfills the estimate*

$$\|g\|_{L^2_t[0,T]L^{2,-a}_x} \le C_0\epsilon + C\epsilon^2 \ , \tag{6.54}$$

where a is a sufficiently large constant and C_0 depends only on the constant of the inequality (6.4).

Proof. Remarking that (where ∇_{ϕ_c} is computed at constant M)

$$J\nabla_{\phi_c}H_{re} = P_c\left[X_N(\phi_c+\phi_d) - X_N(\phi_d)\right] \ , \quad J\nabla_{\phi_c}H_{Nc} = P_cX(\phi_d) - (G+\bar{G}) \ , \tag{6.55}$$

and denoting $\mathcal{R}_\xi := H_{L1} + H_N$, the equation for g takes the form

$$\dot{g} = L(t)g - w^j J\mathcal{A}_j(Y+\bar{Y}) - \mathrm{i}\frac{\partial\bar{Y}}{\partial\xi_k}\frac{\partial\mathcal{R}_\xi}{\partial\bar{\xi}_k} + \mathrm{i}\frac{\partial Y}{\partial\bar{\xi}_k}\frac{\partial\mathcal{R}_\xi}{\partial\xi_k} \tag{6.56}$$

$$+ J\nabla_{\phi_c}H_{L1}(\phi_c) + P_c\left[X_N(\phi_c+\phi_d) - X_N(\phi_d)\right] \tag{6.57}$$

$$+ P_c\left[X(\phi_d) - (G+\bar{G})\right] \ . \tag{6.58}$$

We apply Duhamel formula and estimate the different terms arising. First we consider

$$\int_0^t \mathcal{U}(t,s)w^j J\mathcal{A}_jY(s)\mathrm{d}s \ . \tag{6.59}$$

Using lemma 6.9 and formula (6.40) for Y, it can be rewritten as the sum of finitely many terms of the form

$$\int_0^t \mathcal{U}(t,s)w^jP_c\left[R^{\pm}_{L_c}J\mathcal{A}_j\Phi_\mu + R^{\pm}_{L_c}\Psi_\mu\right]\bar{\xi}^\mu(s)\mathrm{d}s \ , \tag{6.60}$$

with suitable $\Psi_\mu \in \mathcal{W}^\infty$. Then, exploiting (St.2), one has

$$\|(6.60)\|_{L^2_t[0,T]L^{2,-\nu}_x} \preceq \|\xi^\nu\|_{L^2_t}\|\Phi_\nu\|_{L^{2,\nu}_x} \preceq C_2\epsilon^3 \ . \tag{6.61}$$

The estimate of the last term of (6.56) is identical to the same estimate of [BC11], see lemma 7.9, so it is omitted. The terms coming from (6.57) have already been estimated in the proof of lemma 6.6 (see the estimates of I_1, I_2, I_3).

We come to (6.58). To this end remark that one has

$$X(\phi_d(\xi)) = \sum_{\mu,\nu\ :\ \omega\cdot(\nu-\mu)>\Omega} (X_{\mu\nu}\xi^\mu\bar{\xi}^\nu + c.c.) \tag{6.62}$$

(with c.c. denoting the complex conjugated term), while the term subtracted in (6.58) coincides with

$$\sum_{\nu\in\mathcal{K}} (X_{0\nu}\bar{\xi}^\nu + c.c.)\ . \tag{6.63}$$

It follows that, if a term is present in (6.62) but not in (6.63), then it is of the form $X_{\mu\nu}\xi^\mu\bar{\xi}^{\nu+\nu'}$ with $\nu\in\mathcal{K}$. It follows that for such a term

$$\left\| X_{\mu(\nu+\nu')}\xi^\mu\bar{\xi}^{\nu+\nu'} \right\|_{L^2_t[0,T]L^{2,-a}_x} \preceq \left\| X_{\mu(\nu+\nu')} \right\|_{L^{2,-a}_x} \|\xi^\nu\|_{L^2_t[0,T]} |\xi^\mu| \preceq C_2\epsilon^2\ . \tag{6.64}$$

Since the sum is finite the thesis follows. ◻

Lemma 6.11. *Assume* (6.15) *and* (6.16)*, then, provided ϵ is small enough, the following estimate holds*

$$\sum_j \|\eta_j\mathcal{E}_j\|_{L^1_t[0,T]} \le CC_2\epsilon^2 \tag{6.65}$$

The proof of this lemma is almost identical to the proof of Lemma 7.11 of [BC11]. Indeed the only difference is due to the presence of H_{L1}, but the corresponding terms can be estimated by the same methods used in [BC11]. For this reason we omit the proof.

Theorem 6.12. *Assume* (6.15) *and* (6.16) *then, provided ϵ is small enough, one has*

$$\|\phi_c(t)\|_{L^2_t[0,T]L^6_x} \le C(C_2)\epsilon \tag{6.66}$$

$$\omega\cdot\mu > \Omega \implies \|\xi^\mu(t)\|_{L^2_t[0,T]} \le C\sqrt{C_2}\epsilon \tag{6.67}$$

The proof (by standard bootstrap argument) is identical to the proof of Theorem 7.12 of [BC11] and therefore is omitted.

Then also Theorem 6.1 immediately follows.

7 NLS

Consider the scale of real Hilbert spaces $H^{k,l}(\mathbb{R}^n,\mathbb{C})$. We introduce the scalar product in H^0 and the symplectic form as follows:

$$\langle\psi_1;\psi_2\rangle := 2\,\mathrm{Re}\left(\int_{\mathbb{R}^n}\psi_1(x)\bar{\psi}_2(x)dx\right)\ ,\quad \omega(\psi_1;\psi_2) := \langle \mathrm{i}\psi_1;\psi_2\rangle\ , \tag{7.1}$$

(remark that on a real vector space the multiplication by i is not a scalar but a linear operator). The Hamilton equations are given by $\dot{\psi} = -\mathrm{i}\nabla_{\bar{\psi}} H$, where $\nabla_{\bar{\psi}}$ is the gradient with respect of the L^2 scalar product.

The Hamiltonian of the NLS is given by

$$H := \mathcal{P}_0 + H_P \ , \qquad \mathcal{P}_0(\psi) := \int_{\mathbb{R}^3} \bar{\psi}\,(-\Delta\psi) d^3x \ , \quad H_P(\psi) := -\int_{\mathbb{R}^3} \beta(|\psi|^2) d^3x \ ; \tag{7.2}$$

in particular one has $A_0 := -\Delta$. The corresponding Hamilton equations are (1.1).

There are 4 symmetries: Gauge and translations. The operators generating the symmetries are $A_j = -\mathrm{i}\partial_j$, $j = 1, 2, 3$ and $A_4 = \mathbb{1}$, so that one has

$$\mathcal{P}_j(\psi) = -\int_{\mathbb{R}^n} \bar{\psi} \mathrm{i}\partial_j \psi d^n x \ , \quad \mathcal{P}_4(\psi) = \int_{\mathbb{R}^n} |\psi|^2 d^n x \ . \tag{7.3}$$

The construction of the ground state (and the subsequent study) exploits the boost transformation which, given a ground state at rest, puts it in uniform motion.

Definition 7.1. Given $v \in \mathbb{R}^3$, the unitary transformation

$$U(v) : L^2 \ni \psi \mapsto U(v)\psi := e^{-\frac{iv\cdot x}{2}} \psi \in L^2 \ , \tag{7.4}$$

is called the *boost* transformation with velocity v.

Remark 7.2. The boosts form a unitary group parametrized by the velocities. Furthermore, for any fixed v the corresponding boost is a canonical (symplectic) transformation. The boosts have also the remarkable property of conserving the L^p norms.

Having fixed $\mathcal{E} > 0$ and putting $\lambda^4 = -\mathcal{E}$ and $\lambda^j = 0$, for $j = 1, 2, 3$, the equation (3.3) for the ground state, denoted by $b_{\mathcal{E}}$, takes the form (2.1) which has already been discussed. We will denote

$$p_4(\mathcal{E}) := \mathcal{P}_4(b_{\mathcal{E}}) \equiv \int_{\mathbb{R}^n} b_{\mathcal{E}}^2 d^n x \ .$$

A direct computation shows that $\eta_p := U(v) b_{\mathcal{E}}$ is a ground state with parameters

$$\mathcal{P}_4(\eta_p) = p_4(\mathcal{E}) \ , \quad \mathcal{P}_j(\eta_p) = \frac{v_j}{2} p_4(\mathcal{E}), \ j = 1, 2, 3 \tag{7.5}$$

$$\lambda^j = v_j \ , \ j = 1, 2, 3 \ , \quad \lambda^4 = -\left(\mathcal{E} + \frac{|v|^2}{4}\right) \ . \tag{7.6}$$

In order to explicitly perform the computations and to verify all the assumptions it is useful to exploit the existence of the boosts. So fix p_0 and consider η_{p_0} and the decomposition of H^k into $T_{\eta_{p_0}}\mathcal{T}$ and its symplectic orthogonal. Since $U(v)$ maps $b_{\mathcal{E}}$ into η_{p_0}, is linear and symplectic, it also maps $T_{b_{\mathcal{E}}}\mathcal{T}$ to

$T_{\eta_{p_0}}\mathcal{T}$ and $T^{\omega}_{b_{\mathcal{E}}}\mathcal{T}$ to $T^{\omega}_{\eta_{p_0}}\mathcal{T}$. Furthermore it is unitary (and it also conserves all the L^p norms), and therefore it *is particularly convenient to represent $\mathcal{V}$ as $\mathcal{V} = U(v)\mathcal{V}_{\mathcal{E}}$, with $\mathcal{V}_{\mathcal{E}} := T^{\omega}_{b_{\mathcal{E}}}\mathcal{T}$.* **This is what we are now going to do**. *We will also denote by $\Pi_{\mathcal{E}} \equiv \Pi_{b_{\mathcal{E}}}$ the projector on such a space*

Remark 7.3. In such a representation one has that H_{L0} is represented by the restriction to $\mathcal{V}_{\mathcal{E}}$ of $H_{L\mathcal{E}} := \mathcal{P}_0 + d^2H_P(b_{\mathcal{E}})(\psi,\psi) + \mathcal{E}\mathcal{P}_4$. Correspondingly the linear operator $J\nabla H_{L0}$ is equivalent (through U) to the restriction to $\mathcal{V}_{\mathcal{E}}$ of the vector field of H_{LE}, which in turn is the Hamiltonian vector field of $H_{LE}\circ\Pi_{\mathcal{E}}$.

Remark 7.4. Explicitly, one has

$$H_{LE}(\psi) = \frac{1}{2}\langle -\Delta\psi,\psi\rangle + \mathcal{E}\frac{1}{2}\langle\psi,\psi\rangle + d^2H_P(b_{\mathcal{E}})(\psi,\psi) \ , \tag{7.7}$$

or, denoting

$$\psi = \frac{\psi_- + \mathrm{i}\psi_+}{\sqrt{2}} \ , \quad \psi_\pm \in H^{k,l}(\mathbb{R}^n,\mathbb{R}) \ , \tag{7.8}$$

$$H_{LE}(\psi_+,\psi_-) = \frac{1}{2}\langle A_+\psi_+;\psi_+\rangle + \frac{1}{2}\langle A_-\psi_-;\psi_-\rangle \ , \tag{7.9}$$

where

$$A_- := -\Delta + \mathcal{E} - \beta'(b^2_{\mathcal{E}}) - 2\beta''(b^2_{\mathcal{E}})b^2_{\mathcal{E}} \ , \quad A_+ := -\Delta + \mathcal{E} - \beta'(b^2_{\mathcal{E}}) \ , \tag{7.10}$$

one has

$$L_0\begin{bmatrix}\psi_+\\ \psi_-\end{bmatrix} = \begin{bmatrix}-A_-\psi_-\\ A_+\psi_+\end{bmatrix} \tag{7.11}$$

We pass to the verification of the assumptions. (S1-S4) are trivial. The same is true for (P1-P2), (B1-B2). (L1,L2,P3) are well known in this context, while (L3,L4) were assumed expolicitely in sect. 2. (St.3) is by now standard. We come to the other assumptions.

Lemma 7.5. *Assumption (B3) holds.*

Proof. It is clearly enough to verify that, at $\eta_p = b_{\mathcal{E}}$. First remark that, at $b_{\mathcal{E}}$, we have

$$\frac{\partial\eta_p}{\partial p_j} = -\frac{\mathrm{i}}{p_4}x^jb_{\mathcal{E}} \equiv \left(-\frac{1}{p_4}x^jb_{\mathcal{E}},0\right) \ , \ j=1,2,3 \ , \tag{7.12}$$

$$\frac{\partial\eta_p}{\partial p_4} = -\mathcal{E}'\frac{\partial b_{\mathcal{E}}}{\partial\mathcal{E}} \equiv \left(0,-\mathcal{E}'\frac{\partial b_{\mathcal{E}}}{\partial\mathcal{E}}\right) \ , \tag{7.13}$$

so that one gets

$$\begin{aligned}\omega\left(\frac{\partial\eta_p}{\partial p_4};\frac{\partial\eta_p}{\partial p_j}\right) &= -\frac{\mathcal{E}'}{p_4}\left\langle\left(\frac{\partial b_{\mathcal{E}}}{\partial E},0\right);\left(x^jb_{\mathcal{E}},0\right)\right\rangle\\ &= \int_{\mathbb{R}^3} x^jb_{\mathcal{E}}\frac{\partial b_{\mathcal{E}}}{\partial E}dx \ ,\end{aligned}$$

but $b_{\mathcal{E}}\frac{\partial b_{\mathcal{E}}}{\partial E}$ is spherically symmetric, while x^j is skew symmetric, and thus the integral vanishes. ∎

The following Lemma is a minor variant of a Lemma proved by Perelman

Lemma 7.6. *Assumptions (St.1,St.2) hold.*

In appendix B we report its proof.

Corollary 7.7. *Under the assumptions of section 2 Theorem 6.1 holds for the NLS.*

Remark 7.8. In the case of NLS one can easily show that (St.1) and (St.2) hold also if the spaces L^p_x are substituted by the Sobolev spaces $W^{1,p}_x$. As a consequence also the conclusion (6.11) holds with $W^{1,6}_x$ in place of L^6_x.

In the case of NLS, the flow of L_0 is well known to satisfy Strichartz estimates of the form

$$\left\| \mathrm{e}^{tL_0} P_c \phi \right\|_{L^q_t W^{1,r}_x} \preceq \|\phi\|_{H^1} \tag{7.14}$$

$$\left\| \int_0^t \mathrm{e}^{(t-s)L_0} P_c F(s) \mathrm{d}s \right\|_{L^q_t W^{1,r}_x} \preceq \|F\|_{L^{\tilde{q}'}_t W^{1,\tilde{r}'}_x} \tag{7.15}$$

for all admissible pair (q,r), $(\tilde{q},\tilde{r})$, namely pairs fulfilling

$$2/q + 3/r = 3/2, \quad 6 \geq r \geq 2, q \geq 2 \ .$$

As a consequence one can prove the same estimates also for the flow $\mathcal{U}(t,s)$. Using such estimates one gets the following

Theorem 7.9. *Let $\phi(t)$ be a solution of the reduced equations corresponding to NLS. Let ϕ_0 be the initial datum, and assume $\|\phi_0\|_{H^1} = \epsilon$ is small enough. Then there exists ϕ_∞ such that*

$$\lim_{t \to +\infty} \left\| \phi(t) - e^{tL_0} \phi_\infty \right\|_{H^1} = 0 \tag{7.16}$$

Proof. The proof is standard (see e.g. [BC11], Lemma 7.8) and thus it is omitted. Theorem 2.2 is just a reformulation of the above theorem in terms of the original system.

A Proof of theorem 3.14

We recall the idea on which the proof is based in the standard case. The main point is the construction of a suitable coordinate frame in which the actions of the symmetries becomes trivial.

To start with consider the map

$$I \times \mathbb{R}^n \times \mathcal{V}^k \ni (p,q,\phi) \mapsto e^{q^j J A_j} (\eta_p + \Pi_p \phi) \in \mathcal{H}^k \ . \tag{A.1}$$

Lemma A.1. *There exists a mapping $\varphi(u) \equiv (p(u), q(u))$ with the following properties*

1) *$\forall k$ there exists an open neighborhood $\mathcal{U}^{-k} \subset \mathcal{H}^{-k}$ of η_{p_0} such that $\varphi \in C^\infty(\mathcal{U}_{-k}, \mathbb{R}^{2n})$*

2) $e^{-q^j(u)JA_j}u - \eta_{p(u)} \in \Pi_{p(u)}\mathcal{V}^{-k}$.

Proof. Consider the condition 2). It is equivalent to the couple of equations

$$0 = f_l(q,p,u) := \langle e^{-q^j JA_j}u - \eta_p; A_l\eta_p\rangle \equiv \langle u; e^{q^j JA_j}A_l\eta_p\rangle - 2p^j = 0 \ , \tag{A.2}$$

$$0 = g^l(q,p,u) := \langle e^{-q^j JA_j}u - \eta_p; E\frac{\partial\eta_p}{\partial p_l}\rangle \equiv \langle u; e^{q^j JA_j}E\frac{\partial\eta_p}{\partial p_l}\rangle - \langle\eta_p; E\frac{\partial\eta_p}{\partial p_l}\rangle \tag{A.3}$$

Both the functions f and g are smoothing, so we try to apply the implicit function theorem in order to define the functions $q(u)$, $p(u)$. First remark that the equations are fulfilled at $(q,p,u) = (0,p_0,\eta_{p_0})$, then we compute the derivatives of such functions with respect to q^j, p_j and show that they are invertible. We have

$$\frac{\partial f_j}{\partial p_k}\Big|_{(0,p_0,\eta_{p_0})} = \left[\langle u; e^{q^l JA_l}A_j\frac{\partial\eta_p}{\partial p_k}\rangle - 2\delta_j^k\right]_{(0,p_0,\eta_{p_0}))} = -\delta_j^k \ ,$$

where we used

$$\delta_j^k = \frac{\partial}{\partial p_k}\frac{1}{2}\langle\eta_p; A_j\eta_p\rangle = \langle\eta_p; A_j\frac{\partial\eta_p}{\partial p_k}\rangle \ . \tag{A.4}$$

Then we have

$$\frac{\partial f_j}{\partial q_k}\Big|_{(0,p_0,\eta_{p_0})} = \langle\eta_{p_0}; JA_jA_k\eta_{p_0}\rangle = 0 \tag{A.5}$$

by the skewsymmetry of J and property (S1).

We come to g.

$$\frac{\partial g^j}{\partial p_k}\Big|_{(0,p_0,\eta_{p_0})} = \langle\eta_{p_0}; E\frac{\partial^2\eta_{p_0}}{\partial p_j\partial p_k}\rangle - \langle\frac{\partial\eta_{p_0}}{\partial p_k}; E\frac{\partial\eta_{p_0}}{\partial p_j}\rangle - \langle\eta_{p_0}; E\frac{\partial^2\eta_{p_0}}{\partial p_j\partial p_k}\rangle \tag{A.6}$$

which vanishes by (H6). Finally we have

$$\frac{\partial g^j}{\partial q^k}\Big|_{(0,p_0,\eta_{p_0})} = \langle A_k\eta_{p_0}; \frac{\partial\eta_{p_0}}{\partial p_j}\rangle = \delta_k^j$$

Therefore the implicit function theorem applies and gives the result. □

Corollary A.2. *Any function $u \in \mathcal{H}^k$ in a neighborhood of η_{p_0} can be uniquely represented as*

$$u = e^{q^j JA_j}\left(\eta_p + \Pi_p\phi\right) \ , \tag{A.7}$$

with $(q,p,\phi) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{V}^k$ smoothly dependent on u.

Remark that $p(u) \neq \mathcal{P}(u)$.

Lemma A.3. *Fix $\phi \in \mathcal{V}^l$, and let $X \in T_{i(\phi)}\mathcal{S} \cap \mathcal{H}^k$. Assume $k + l \geq d_A$, then there exist $Q \equiv (Q^1, ..., Q^n) \in \mathbb{R}^n$ and $\Phi \in \mathcal{H}^{\min\{k,l-d_A\}}$ such that*

$$X = Q^j JA_j i(\phi) + i_*\Phi \ . \tag{A.8}$$

Proof. We write explicitly the formula (A.8) and show how to solve it for Q and Φ:

$$X = Q^j JA_j(\eta_p + \Pi_p\phi) + \left(\frac{\partial \eta_p}{\partial p_j} + \frac{\partial \Pi_p}{\partial p_j}\phi\right) \langle \nabla p_j; \Phi\rangle + \Pi_p\Phi \ . \tag{A.9}$$

Apply to such a formula the operator $\widetilde{\Pi_p}^{-1}\Pi_p$ (recall that $\widetilde{\Pi}_p : \Pi_p\mathcal{H}^i \to \mathcal{V}^i$ is an isomorphism) getting

$$\Phi = \widetilde{\Pi_p}^{-1}\Pi_p X - Q^j\widetilde{\Pi_p}^{-1}\Pi_p JA_j(\eta_p + \Pi_p\phi) - \widetilde{\Pi_p}^{-1}\Pi_p\left(\frac{\partial \eta_p}{\partial p_j} + \frac{\partial \Pi_p}{\partial p_j}\phi\right) \langle \nabla p_j; \Phi\rangle \ . \tag{A.10}$$

Taking the scalar product with ∇p_k we get

$$\begin{aligned}\langle \nabla p_k; \Phi\rangle = &\left\langle \nabla p_k; \widetilde{\Pi_p}^{-1}\Pi_p X\right\rangle - Q^j\left\langle \nabla p_k; \widetilde{\Pi_p}^{-1}\Pi_p JA_j(\eta_p + \Pi_p\phi)\right\rangle \\ &- \left\langle \nabla p_k; \widetilde{\Pi_p}^{-1}\Pi_p\left(\frac{\partial \eta_p}{\partial p_j} + \frac{\partial \Pi_p}{\partial p_j}\phi\right)\right\rangle \langle \nabla p_j; \Phi\rangle\end{aligned} \tag{A.11}$$

By formula (3.21) and by remark 3.8 one has

$$\left\langle \nabla p_k; \widetilde{\Pi}_{p_{-1}}\Pi_p X\right\rangle = -\sum_l M_k^l\langle A_l\phi; X\rangle + \text{smoothing function} \ ,$$

which is well defined under the assumptions of the lemma. Now,

$$\begin{aligned}\left\langle \nabla p_k; \widetilde{\Pi_p}^{-1}\Pi_p JA_j\Pi_p\phi\right\rangle &= -\sum_l M_k^l\langle A_l\phi; JA_j\phi\rangle + \text{smoothing function} \\ &= \text{smoothing function} \ .\end{aligned}$$

Furthermore the coefficient of $\langle \nabla p_j; \Phi\rangle$ at r.h.s. of (A.11) is small (and smoothing) if ϕ is small enough. Thus one can solve (A.11) and compute $\langle \nabla p_j; \Phi\rangle$ as a function of well defined objects and of Q^j.

Take the scalar product of (A.9) with $E\frac{\partial \eta_p}{\partial p_l}$, getting

$$\begin{aligned}\langle X; E\frac{\partial \eta_p}{\partial p_l}\rangle &= Q^j\left(\langle E\frac{\partial \eta_p}{\partial p_l}; JA_j\eta_p\rangle + \langle E\frac{\partial \eta_p}{\partial p_l}; JA_j\Pi_p\phi\right) \\ &\quad + \left(\langle E\frac{\partial \eta_p}{\partial p_l}; \frac{\partial \eta_p}{\partial p_j}\rangle + \langle E\frac{\partial \eta_p}{\partial p_l}; \frac{\partial \Pi_p}{\partial p_j}\phi\rangle\right)\langle \nabla p_j; \Phi\rangle \\ &= Q^j(-\delta_j^l - \langle\frac{\partial \eta_p}{\partial p_l}; A_j\Pi_p\phi\rangle) + \langle \nabla p_j; \Phi\rangle\langle E\frac{\partial \eta_p}{\partial p_l}; \frac{\partial \Pi_p}{\partial p_j}\phi\rangle \ .\end{aligned}$$

Substitute the expression we got for $\langle \nabla p_j; \Phi\rangle$, and then it is immediate to see that it is possible to compute the Q_j's, and thus also $\langle \nabla p_j; \Phi\rangle$, and use (A.10) to get Φ. $\square$

Lemma A.4. *Take $\phi \in \mathcal{V}^k$ with k large enough, then the following formula holds*

$$X_H(e^{q^j JA_j} i(\phi)) = Q^l(\phi) e^{q^j JA_j} JA_l i(\phi) + e^{q^j JA_j} i_* X_{H_r}(\phi) \ . \tag{A.12}$$

Furthermore there exists a matrix $\tilde{M}_j^l = \delta_j^l + \hat{M}_j^l$ with $\hat{M}_j^l$ smoothing functions, such that

$$Q^l = \tilde{M}_j^l \mathrm{d}H \frac{\partial \eta_{p_0}}{\partial p_j} \ . \tag{A.13}$$

Proof. First, the vector field X_H is equivariant, i.e.

$$X_H(e^{q^j JA_j}\phi) = e^{q^j JA_j} X_\phi(\phi) \ ,$$

thus it is enough to verify the formula for $q^j = 0$. Furthermore $X_H(i(\phi)) \in T_{i(\phi)}\mathcal{S}$ thus, by the preceding lemma it admits the representation

$$X_H = Q^l JA_l i(\phi) + i_*\Phi \ , \tag{A.14}$$

and remark that, for any choice of l, one has

$$\omega(JA_l i(\phi); i_*\Psi) = \langle A_l i(\phi); i_*\Psi\rangle = 0 \ , \quad \forall \Psi \in \mathcal{V}$$

since this is the condition ensuring that $i_*\Psi \in T_{i(\phi)}\mathcal{S}$.

Remark that (by lemma A.3), at such points, any vector $U \in \mathcal{H}^k$ admits the representation

$$U = \alpha^l JA_l i(\phi) + i_*\Psi + \beta^j \frac{\partial \eta_p}{\partial p_j} \ ; \tag{A.15}$$

we insert such a representations in the definition of the vector field X_H. Obtaining

$$\begin{aligned} \mathrm{d}HU &= \alpha^l \mathrm{d}HJA_l i(\phi) + \mathrm{d}H i_*\Psi + \beta^j \mathrm{d}H \frac{\partial \eta_p}{\partial p_j} \\ &= \omega(X_H; U) = \omega(Q^l JA_l i(\phi) + i_*\Phi; \alpha^l JA_l i(\phi) + i_*\Psi + \beta^j \frac{\partial \eta_p}{\partial p_j}) \ , \end{aligned}$$

which, exploiting the invariance of H and (A.15), gives

$$\mathrm{d}H i_*\Psi + \beta^j \mathrm{d}H \frac{\partial \eta_p}{\partial p_j} = Q^l \beta_j \omega(JA_l i(\phi); \frac{\partial \eta_p}{\partial p_j}) + \omega(i_*\Phi; i_*\Psi) + \beta^j \omega(i_*\Phi; \frac{\partial \eta_p}{\partial p_j}) \ . \tag{A.16}$$

Taking $\beta^j = 0$ we get $\mathrm{d}(i^*H)\Psi = i^*\omega(\Phi; \Psi)$, which shows that $\Phi = X_{H_r}$. To get the formula for the Q's take $\Phi = 0$ and all the β's equal to zero but one. Thus we get

$$\begin{aligned} \mathrm{d}H \frac{\partial \eta_p}{\partial p_j} = Q^l \omega(JA_l i(\phi); \frac{\partial \eta_p}{\partial p_j}) &= Q^l \langle A_l(\eta_p + \Pi_p\phi); \frac{\partial \eta_p}{\partial p_j}\rangle \\ &= Q^l \left(\delta_l^j + \langle A_l \Pi_p \phi; \frac{\partial \eta_p}{\partial p_j}\rangle\right) \end{aligned}$$

which gives the thesis. □

From this Lemma the thesis of the theorem immediately follows.

B Proof of Perelman's Lemma 7.6

First we transform the equation $\dot{\phi}_c = L(t)\phi_c$ to a more suitable form. To this end we make the transformation

$$\phi = e^{q^j(t)JA_j}\tilde{\phi} \ , \quad \dot{q}^j = w^j \ , \quad q^j(0) = 0 \ , \tag{B.1}$$

which gives

$$\frac{d}{dt}\tilde{\phi} = P_c(t)H(t)\tilde{\phi} - \tilde{R}\tilde{\phi} \ , \tag{B.2}$$

where

$$H(t) := J[A_0 + \tilde{V}(t) + EA_4] \tag{B.3}$$

and

$$\begin{aligned} &P_c(t) := e^{-q^J A_j} P_c e^{q^j JA_j} \ , \quad \tilde{V}(t) := e^{-q^j JA_j} V_0 e^{q^j JA_j} \ , \\ &\tilde{R} := w^j \left[P_c(t) - \mathbb{1}\right] JA_j P_c(t) + w^j P_c(t) JA_j \left[P_c(t) - \mathbb{1}\right] \ . \end{aligned}$$

Explicitly the operators $\tilde{V}(t)$ and $P_c(t)$ can be computed by remarking that, since $e^{q^j JA_j}$ is canonical and unitary for any fixed time, one has

$$\begin{aligned} d^2 H_P(b_{\mathcal{E}})(e^{q^j JA_j}\tilde{\phi}, e^{q^j JA_j}\tilde{\phi}) = \frac{1}{2}\left\langle V_0 e^{q^j JA_j}\tilde{\phi}; e^{q^j JA_j}\tilde{\phi}\right\rangle = \frac{1}{2}\langle \tilde{V}\tilde{\phi}; \tilde{\phi}\rangle \\ = d^2 H_P(e^{-q^j JA_j} b_{\mathcal{E}})(\tilde{\phi}, \tilde{\phi}) \ . \end{aligned}$$

Thus the projector $P_c(t)$ is the projector on the continuous spectrum of $H(t)$. From this and the fact that $\mathrm{e}^{q^j JA_j}\tilde{\phi} \in P_c\mathcal{V}$, it follows in particular that, for any time t, one has $P_c(t)\tilde{\phi}(t) = \tilde{\phi}(t)$. Remark also that one has

$$\begin{aligned} \tilde{R} &= w^j \left[P_c(t) - \mathbb{1}\right] JA_j P_c(t) + w^j P_c(t) JA_j \left[P_c(t) - \mathbb{1}\right] \\ &= -w^j [P_d(t) JA_j P_c(t) + P_c(t) JA_j P_d(t)] \end{aligned} \tag{B.4}$$

which therefore is a small smoothing operator. **Omitting tildes** one has the explicit formula

$$\begin{aligned} V(t)\phi = -\beta'(b^2_{\mathcal{E}}(x - \mathbf{q}(t)))\phi \\ -\beta''(b^2_{\mathcal{E}}(x - \mathbf{q}(t)))2\,\mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}q^4(t)} b_{\mathcal{E}}(x - \mathbf{q}(t))\phi\right) \mathrm{e}^{\mathrm{i}q^4(t)} b_{\mathcal{E}}(x - \mathbf{q}(t)) \ . \end{aligned} \tag{B.5}$$

Here and below we denote by $\mathbf{q} \in \mathbb{R}^3$ the vector with components q^j, $j = 1, 2, 3$.

We work on the equation

$$\dot{\phi} = H(t)\phi + R(t)\phi \ , \tag{B.6}$$

following almost literally the proof given by Perelman. With a slight abuse of notation we will here denote by $\mathcal{U}(t, s)$ the evolution operator of such an equation. Remark that, fro the fact that $L(t)$ leaves $P_c\mathcal{V}^k$ invariant, one has

$$P_c(t)\mathcal{U}(t, s) = \mathcal{U}(t, s)P_c(s) \ . \tag{B.7}$$

First we have the following proposition (which follows from proposition 1.1 of [Per04] and the remark that $\mathrm{e}^{q^j JA_j}$ conserves all the L^p_x norms)

Proposition B.1. *There exists ϵ_0 such that, provided $|w^j| = |\dot{q}^j(t)| < \epsilon_0$, then one has*

$$\sup_{a\in\mathbb{R}^3,\ t\in\mathbb{R},\ s\in\mathbb{R}} \left(\left\| \langle x-a\rangle^{-\nu} \mathrm{e}^{H(t)s} P_c(t)\phi \right\|_{L^2} \langle s\rangle^{3/2} \right) \preceq \left(\|\phi\|_{L^2} + \|\phi\|_{L^1} \right) . \quad \text{(B.8)}$$

We are going to prove the following local decay estimate from which the Strichartz type inequalities (6.2) and (6.3) follow.

Lemma B.2. *The evolution operator $\mathcal{U}$ satisfies*

$$\left\| \langle x-a\rangle^{-\nu} \mathcal{U}(t,s) P_c(s)\phi \right\|_{L^2} \preceq \frac{\|\phi\|_{L^2} + \|\phi\|_{L^1}}{\langle t-s\rangle^{3/2}} , \quad \text{(B.9)}$$
$$\forall t \geq s , \quad \forall a \in \mathbb{R}^3 .$$

Proof. It is clearly sufficient to work with $s = 0$. We will make use of the following Duhamel formula

$$\phi(t) = \mathrm{e}^{H(t)t}\phi_0 + \int_0^t ds \mathrm{e}^{H(t)(t-s)}[H(s) - H(t)]\phi(s) + \int_0^t ds \mathrm{e}^{H(t)(t-s)} R(s)\phi(s) .$$

Applying $P_c(t)$ and iterating once the formula one gets $\phi(t) = I_1 + I_2 + I_3 + I_4 + I_5$, where

$$\begin{aligned}
I_1 &= \mathrm{e}^{H(t)t} P_c(t)\phi_0 \quad I_2 = \int_0^t ds\, \mathrm{e}^{H(t)(t-s)} P_c(t) R(s)\phi(s) \\
I_3 &= \int_0^t ds\, \mathrm{e}^{H(t)(t-s)} P_c(t)[H(s) - H(t)] \mathrm{e}^{H(s)s} P_c(s)\phi_0 , \\
I_4 &= \int_0^t ds \int_0^s ds_1 \mathrm{e}^{H(t)(t-s)} P_c(t)[H(s) - H(t)] \mathrm{e}^{H(s)(s-s_1)} P_c(s) R(s_1)\phi(s_1) , \\
I_5 &= \int_0^t ds \int_0^s ds_1 \mathrm{e}^{H(t)(t-s)} P_c(t)[H(s) - H(t)] \mathrm{e}^{H(s)(s-s_1)} P_c(s)[H(s_1) - H(s)]\phi(s_1) .
\end{aligned}$$

The only nontrivial estimate is that of I_5. We start by the others and then we concentrate on I_5,

The estimate of I_1 is an immediate consequence of proposition B.1. Define

$$m(t) := \sup_{a\in\mathbb{R}^3,\ 0\leq\tau\leq t} \left(\left\| \langle x-a\rangle^{-\nu} \phi(t) \right\|_{L^2} \langle \tau\rangle^{3/2} \right) , \quad \text{(B.10)}$$

and, in order to estimate I_2 remark that, for $p = 1, 2$, one has

$$\|R(s)\phi(s)\|_{L^p} \leq \left\| \langle x - \mathbf{q}(s)\rangle^N R(s)\phi(s) \right\|_{L^2} \preceq \epsilon \left\| \langle x - \mathbf{q}(s)\rangle^{-\nu} \phi(s) \right\|_{L^2} \leq \epsilon \frac{m(s)}{\langle s\rangle^{3/2}}$$

Substituting in I_2 one gets

$$\begin{aligned}
\left\| \langle x-a\rangle^{-\nu} I_2(t) \right\|_{L^2} &\preceq \int_0^t ds \frac{1}{\langle t-s\rangle^{3/2}} \left(\|R(s)\phi(s)\|_{L^1} + \|R(s)\phi(s)\|_{L^2} \right) \\
&\preceq \epsilon m(t) \int_0^t ds \frac{1}{\langle t-s\rangle^{3/2}} \frac{1}{\langle s\rangle^{3/2}} \preceq m(t) \frac{1}{\langle t\rangle^{3/2}}
\end{aligned}$$

The estimate of I_4 is similar. For estimating I_3, first remark that $H(s)-H(t) = V(s) - V(t)$, and thus, by (B.5), for any function ϕ, one has

$$|[H(s) - H(t)]\,\phi| \preceq \left|\langle x - \mathbf{q}(s)\rangle^{-N}\phi\right| + \left|\langle x - \mathbf{q}(s)\rangle^{-N}\phi\right| \ , \ \forall N \ .$$

So, we have

$$\begin{aligned}
&\left\|\langle x-a\rangle^{-\nu} I_3(t)\right\|_{L^2} \\
&\preceq \int_0^t \frac{ds}{\langle t-s\rangle^{3/2}} \left[\left\|\langle x - \mathbf{q}(s)\rangle^{-\nu} \mathrm{e}^{H(s)s} P_c(s)\phi_0\right\|_{L^2} + \left\|\langle x - \mathbf{q}(t)\rangle^{-\nu} \mathrm{e}^{H(s)s} P_c(s)\phi_0\right\|_{L^2}\right] \\
&\preceq \int_0^t \frac{ds}{\langle t-s\rangle^{3/2}\langle s\rangle^{3/2}} (\|\phi_0\|_{L^1} + \|\phi_0\|_{L^2}) \preceq \frac{1}{\langle t\rangle^{3/2}} (\|\phi_0\|_{L^1} + \|\phi_0\|_{L^2}) \ .
\end{aligned}$$

We come to I_5. Here the key remark is that (with a slight abuse of notation)

$$|V(s_1) - V(s_2)| \preceq \epsilon^{1/2} \langle x - \mathbf{q}(s_1)\rangle^{-N} \quad , \quad \forall N \tag{B.11}$$

and all s_1, s_2 with $|s_1 - s_2| \preceq \epsilon^{-1/2}$.

Consider the two cases $t \leq 4\epsilon^{-1/2}$ and $t \geq 4\epsilon^{-1/2}$. Exploiting (B.11) one easily gets that in the first case

$$\left\|\langle x-a\rangle^{-\nu} I_5(t)\right\|_{L^2} \preceq \epsilon^{1/2} \frac{m(t)}{\langle t\rangle^{3/2}} \ .$$

In the second case $t \geq 4\epsilon^{-1/2}$, split the interval of integration of s into three parts, accordingly define

$$I_{51} = \int_0^{\epsilon^{-1/2}} ds \ , \ I_{52} = \int_{\epsilon^{-1/2}}^{t-\epsilon^{-1/2}} ds \ , \ I_{53} = \int_{t-\epsilon^{-1/2}}^{t} ds \ .$$

The term I_{51} is estimated exploiting the fact that in the considered interval $V(s) - V(s_1)$ fulfill the estimate (B.11). Thus one gets

$$\left\|\langle x-a\rangle^{-\nu} I_{51}(t)\right\|_{L^2} \preceq \epsilon^{1/2} \frac{m(t)}{\langle t\rangle^{3/2}} \ .$$

Similarly I_{53} is estimated using the fact that in such an interval $V(s) - V(t)$ fulfill the estimate (B.11), and thus it gives the same contribution as I_{51}. Finally concerning I_{52}, one has

$$\begin{aligned}
\left\|\langle x-a\rangle^{-\nu} I_{52}(t)\right\|_{L^2} \preceq= &\int_{\epsilon^{-1/2}}^{t-\epsilon^{-1/2}} ds \int_0^s ds_1 \frac{1}{\langle t-s\rangle^{3/2}} \frac{1}{\langle s-s_1\rangle^{3/2}} \frac{m(t)}{\langle s_1\rangle^{3/2}} \\
&\preceq m(t) \int_{\epsilon^{-1/2}}^{t-\epsilon^{-1/2}} ds \frac{1}{\langle t-s\rangle^{3/2}} \frac{1}{\langle s\rangle^{3/2}} \preceq \frac{\epsilon^{1/4} m(t)}{\langle t\rangle^{3/2}} \ .
\end{aligned}$$

Collecting all the results one gets

$$m(t) = \sup\left(\langle t\rangle^{3/2} \left\|\langle x-a\rangle^{-\nu}\phi(t)\right\|_{L^2}\right) \preceq \|\phi_0\|_{L^2} + \|\phi_0\|_{L^1} + \epsilon^{1/4} m(t) \ , \tag{B.12}$$

from which the thesis immediately follows. □

End of the proof of lemma 7.6. Consider the following Duhamel formulae

$$\mathcal{U}(t,0)P_c(0)\phi_0 = \mathrm{e}^{tL_c}P_c(0)\phi_0 + \int_0^t \mathrm{d}s\mathrm{e}^{L_c(t-s)}(V(s)+R(s))\mathcal{U}(s,0)P_c(0)\phi_0 \tag{B.13}$$

$$\mathcal{U}(t,0)P_c(0)\phi_0 = P_c(t)\mathrm{e}^{tL_c}P_c(0)\phi_0 + \int_0^t \mathrm{d}s\mathcal{U}(t,s)P_c(s)(V(s)+R(s))\mathcal{U}(s,0)\mathrm{e}^{L_c s}\phi_0 \ . \tag{B.14}$$

Inserting the second one in the integral of the first one one gets that the quantity to be estimated is the sum of three integrals, which can be easily estimated using (B.9) and the fact that $\mathrm{e}^{L_c t}$ fulfills the Strichartz estimate (6.2) as proved e.g. in [Cuc01] or [Per04].

The retarded estimate (6.3) can be deduced from (6.2) by reproducing exactly the argument by Keel and Tao.

The verification of (St.2) is a small variant and is omitted. □

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